Remark 1: It is possible to prove the following stronger result (useful in particular if $|k|$ is very large):

$$\sup_{k\in\mathbb{Z}} \max\left(1, \left(\frac{k-an}{\sqrt{n}}\right)^2\right) \left| \sigma\sqrt{n}\,\mathbb{P}(Z_n=k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{k-an}{\sigma\sqrt{n}}\right)^2\right)\right| \xrightarrow[n\to\infty]{} 0$$

See Principles of Random Walk, Chapter 7, by Spitzer

Remark 2: When $\sigma^2=\infty$, it is possible to extend the Local Limit theorem when the law of Z_1 belongs to the domain of attraction of a stable law of index $\alpha\in(0,2)$

If $\mathbb{P}(Z_1<-1)=0$, this is equivalent to the fact that $\mathbb{P}(Z_1\geq k) = \frac{L(k)}{k^\alpha}$ where L satisfies

$$\frac{L(\lfloor an\rfloor)}{L(n)} \xrightarrow[n\to\infty]{} 1$$

for every $a>0$

6) Applications of the Local Limit Theorem

Recall that $W_n = X_1+\cdots+X_n$ with $\mathbb{P}(X_1=i)=\mu(i+1)$, $i\geq -1$ and that $\mathbb{E}[X_1] = \sum_{i\geq -1} i\mu(i+1) = \sum_{i\geq 0} i\mu(i) - 1$

Assume that μ is critical and has finite variance σ^2. That $\mathbb{E}[X_1]=0$, $\mathbb{E}[X_1^2] = \sum_{i\geq-1} i^2\mu(i+1) = \sum_{i\geq 0}(i-1)^2\mu(i) = \sum_{i\geq 0} i^2\mu(i) - 1$
$= \sigma^2$

Definition: We say that μ is aperiodic if the maximal integer $h \geq 1$ such that
$$\{i \geq 1 ; \mu(i) > 0\} \subset h\mathbb{Z}$$
is $h=1$. Or, equivalently, that $\gcd\{i \geq 1 ; \mu(i) > 0\} = 1$.

Lemma: Assume that μ is aperiodic and $\mu(0) > 0$. Then $\exists N$ s.t $n \geq N \Rightarrow \mathbb{P}_\mu(|T| = n) > 0$

Proof: by aperiodicity, we can find i and j s.t $\mu(i) > 0$, $\mu(j) > 0$ and $\gcd(i,j) = 1$.

Take $N = ij$.

Fix $n \geq N+1$. Let $0 \leq b \leq i-1$ be the integer such that $bj \equiv n-1 \mod i$ (possible since $\gcd(i,j)=1$)

Hence there exists $a \in \mathbb{Z}$ such that $n-1-bj = ai$.

But $bj < ij \leq n-1$. Hence $a \geq 0$.

Thus $n-1 = ai + bj$ with $a \geq 0$, $b \geq 0$.

Hence with positive probability a GW tree has a vertices with i children, b vertices with j children and all the other vertices 0 children. But such a tree has n vertices in total. Hence $\mathbb{P}_\mu(|T| = n) > 0$. $\boxtimes$

Remark: If μ aperiodic, then X_1 is aperiodic (in the sense defined in lecture 5).

(Indeed, if $\text{Support}(X_1) \subset c + h\mathbb{Z}$, then $\{i \geq 1 ; \mu(i) > 0\} \subset c - 1 + h\mathbb{Z}$ and $\mu(i) > 0, \mu(j) > 0 \Rightarrow h \mid i - j$)

(*) Application of the Local Limit Theorem to the study of the size of GW trees

Assume that μ is critical, aperiodic, with finite variance.

We have seen that $\mathbb{P}_\mu(|T| = n) = \frac{1}{n} \mathbb{P}(W_n = -1)$.

By the local limit theorem, $\mathbb{P}(W_n = -1) = \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{1}{2}\left(-\frac{1}{\sigma\sqrt{n}}\right)^2} + \frac{\varepsilon(k,n)}{\sigma\sqrt{n}}$.

with $\sup_{k \in \mathbb{Z}} \varepsilon(k,n) \xrightarrow[n \to \infty]{} 0$. Hence $\mathbb{P}(W_n = -1) \underset{n \to \infty}{\sim} \frac{1}{\sigma\sqrt{2\pi n}}$.

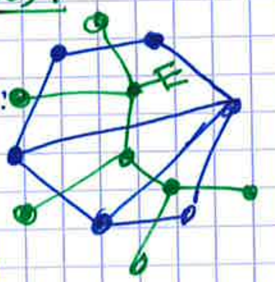
Hence $\boxed{\mathbb{P}_\mu(|T|=n) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \times \frac{1}{n^{3/2}}}$

(in particular, note that $\mathbb{E}_\mu[|T|] = \infty$, which is consistent with the fact that $\mathbb{E}_\mu[|T|] = \mathbb{E}\left[\sum_{n\geq 0} Z_n\right] = \sum_{n\geq 0} \mathbb{E}[Z_n]$

$$= \sum_{n \geq 0} 1 = \infty \quad)$$

4) Application to counting dissections.

Recall the construction of the dual tree $T(D)$ of a dissection D:



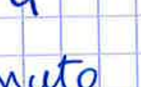
If D_n is a uniform dissection of P_n, we have seen that the law of $T(D_n)$ is $\mathbb{P}_\mu(\cdot \mid \lambda(T) = n-1)$ where $\mu(0) = \frac{1-2c}{1-c}$, $\mu(1) = 0$, $\mu(i) = c^{i-1}$ for $i \geq 2$.

We choose c so that μ is critical: $c = 1 - \frac{\sqrt{2}}{2}$

then $\mu(0) = 2-\sqrt{2}$, $\mu(1) = 0$, $\mu(i) = \left(\frac{2-\sqrt{2}}{2}\right)^{i-1}$ for $i \geq 2$.

Hence:

$$\frac{1}{|\mathbb{D}_n|} = \mathbb{P}_\mu\Big(T = $$



$$\Big|\ \lambda(T) = n-1\Big) = \frac{\mu(n-1)\cdot\mu(0)^{n-1}}{\mathbb{P}_\mu(\lambda(T) = n-1)}$$

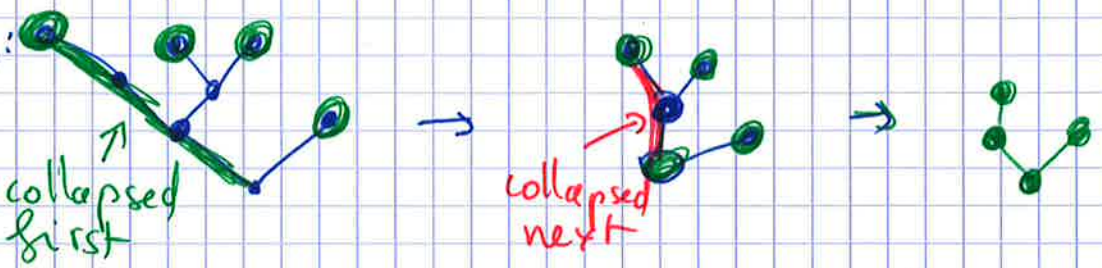
Hence $|\mathbb{D}_n| = \frac{(2-\sqrt{2})^3}{4} \cdot \frac{1}{(3-2\sqrt{2})^n} \cdot \mathbb{P}_\mu(\lambda(T) = n-1)$.

How can we estimate $\mathbb{P}_\mu(\lambda(T) = n-1)$?

We use a construction due to Rizzolo (2011!) which maps a GW tree with n leaves to a GW tree with n vertices:

Start with a tree T with n leaves. Then, iteratively, "collapse" the edges of T that lead to the left-most leaf! to get $\tilde{T}$

For example:



Rizzolo (2011) showed that if T is a GW_μ tree, then $\tilde{T}$ is a $GW_{\tilde{\mu}}$ tree, and :

that μ critical with variance σ^2

$\Rightarrow \tilde{\mu}$ critical with variance $\sigma^2/\mu(0)$.

Hence $\mathbb{P}_\mu(\lambda(T) = n-1)$

$$= \mathbb{P}_\mu(|\tilde{T}| = n-1)$$

$$= \mathbb{P}_{\tilde{\mu}}(|T| = n-1) \sim \frac{1}{2\pi\sigma^2/\mu(0)} \times \frac{1}{n^{3/2}}.$$

We conclude by putting the pieces together that

$$\boxed{|\mathbb{P}_n| \sim \frac{1}{4\sqrt{\pi}} \cdot \sqrt{99\sqrt{2}-140}\,(3+2\sqrt{2})\, n^{-3/2}}$$

(*) Number of children of the root.

We know that $\mathbb{P}_\mu(k_\varnothing = i) = \mu(i)$.

But what about $\mathbb{P}_\mu(k_\varnothing = i \mid |T| = n)$ as $n \to \infty$?

The coding by the Lukasiewicz path gives that

$$\mathbb{P}_\mu(k_\varnothing = i \mid |T| = n) = \mathbb{P}(X_1 = i-1 \mid \zeta_1 = n)$$

$$= \frac{\mathbb{P}(X_1 = i-1, \zeta_1 = n)}{\mathbb{P}(\zeta_1 = n)}$$

$$= \frac{1}{\mathbb{P}(\zeta_1 = n)} \cdot \mu(i) \cdot \mathbb{P}(\zeta_i = n-1)$$

$$= i\mu(i) \times \frac{n}{n-1} \times \frac{\mathbb{P}(W_{n-1} = -i)}{\mathbb{P}(W_n = -1)} \quad \text{since } \mathbb{P}(\zeta_i = k) = \frac{i}{k}\mathbb{P}(W_{k-1}$$

$$\xrightarrow[n\to\infty]{} i\mu(i) \text{ by the Local Limit Theorem.}$$

V Local limits of Galton-Watson trees.

Here μ is an offspring distribution on $\mathbb{Z}_+$. We assume that μ is aperiodic.

1) Critical and finite variance case.

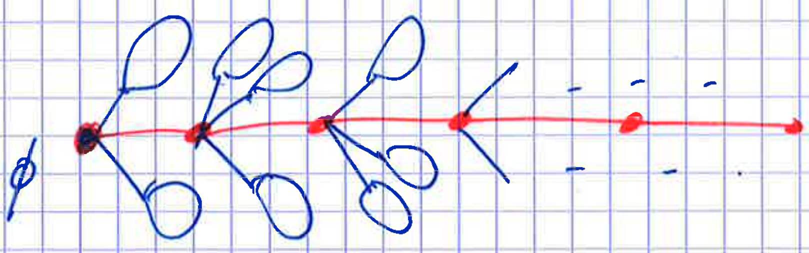
Assume that $\sum_{i \geq 0} i\mu(i) = 1$ and $\sigma^2 = \sum_{i \geq 0} i^2 \mu(i) - 1 \in (0, \infty)$.

Let T_n be a random tree with law $GW_\mu(\cdot \mid |T| = n)$.

Goal: T_n converges "locally" to an ∞ tree. T_∞

Define T_∞ as the following ∞ tree:

- There are 2 types of vertices: normal and mutant
- The number of children of normal vertices have law μ, and they are normal
- The number of children of mutant vertices have law $\bar{\mu}$ defined by $\bar{\mu}(i) = i\mu(i)$, and all are normal, except one which is mutant and which is chosen uniformly among the children:



T_∞

For $t \in \mathbb{T}$, set $Z_k(t) = \{u \in \underline{t} \,;\, |u| = k\}$ and $[t]_k = \{u \in \underline{t} \,;\, |u| \leq k\}$.

Prop For every finite tree t_0

$$\mathbb{P}_\mu([T_\infty]_k = \underline{t}_0) = Z_k(t_0)\, \mathbb{P}_\mu([T]_k = t_0)$$

Proof: exercise.

Thm Assume that μ is critical, aperiodic, has finite variance. For every finite tree τ_0 and $k \geq 0$,

$$\mathbb{P}([T_n]_k = \tau_0) \xrightarrow[n \to \infty]{} \mathbb{P}([T_\infty]_k = \tau_0)$$

We say that T_n converges locally in distribution to T_∞.

Remarks

- The local convergence may be defined by a metric
- If a distinguished point is not given, one can choose it uniformly at random, in which case one speaks of local <u>weak</u> convergence.

Proof: By the proposition, it is enough to check that for every tree t_0 of height k we have

$$\mathbb{P}([T_n]_k = \tau_0) \longrightarrow Z_k(t_0)\, \mathbb{P}_\mu([T]_k = t_0).$$

But

$$\mathbb{P}([T_n]_k = \tau_0) = \mathbb{P}_\mu([T]_k = \tau_0) \times \frac{\mathbb{P}_{\mu, Z_k(\tau_0)}(|F| = n - |t_0|)}{\mathbb{P}_\mu(|T| = n)}$$

$$= \mathbb{P}_\mu([T]_k = \tau_0) \times \frac{Z_k(\tau_0)}{n - |\tau_0|} \times n \times \frac{\mathbb{P}(W_{n-|t_0|} = -Z_k)}{\mathbb{P}(W_n = -1)}$$

(Local limit theorem) $\xrightarrow[n \to \infty]{}$

This completes the proof. ☒

Remark: the theorem is true even if $\sigma^2 = \infty$, indeed it is known that $\forall m, k \in \mathbb{Z}$,

$$\frac{\mathbb{P}(W_{n-m} = -k)}{\mathbb{P}(W_n = -1)} \xrightarrow[n \to \infty]{}$$

(Strong ratio limit theorem for random walks)