Lecture 5 We start by some remarks on the simply generated tree model

Proposition Fix $a, c > 0$ and set $\tilde{\lambda}(i) = a \cdot c^i \cdot \lambda(i)$. Then $\mathbb{P}_\lambda^{(n)} = \mathbb{P}_{\tilde{\lambda}}^{(n)}$.

Proof: Set $\tilde{w}(\underline{t}) = \prod_{u \in \underline{t}} \tilde{\lambda}(k_u)$ and $\tilde{Z}_n = \sum_{|\underline{t}|=n} \tilde{w}(\underline{t})$.

Then for $\underline{t} \in \mathbb{T}_n$,

$$\mathbb{P}_{\tilde{\lambda}}^{(n)}(\underline{t}) = \frac{1}{\tilde{Z}_n} \prod_{u \in \underline{t}} (a \cdot c^{k_u} \lambda(k_u)) = \frac{1}{\tilde{Z}_n} a^n c^{n-1} w(\underline{t})$$

$$= \mathbb{P}_\lambda^{(n)}(\underline{t}) \cdot \frac{Z_n}{\tilde{Z}_n} a^n c^{n-1}$$

Since $\mathbb{P}_{\tilde{\lambda}}^{(n)}$ and $\mathbb{P}_\lambda^{(n)}$ are probability measures, summing over all $\underline{t} \in \mathbb{T}_n$, we get $\frac{Z_n}{\tilde{Z}_n} a^n c^{n-1} = 1$ and the conclusion follows. $\blacksquare$

Remark If there exists $c > 0$ such that $\sum_{i \geq 0} \lambda(i) c^i < \infty$, set

$$\tilde{\lambda}(i) = \frac{\lambda(i) c^i}{\sum_{i \geq 0} \lambda(i) c^i}.$$

Then for every $\underline{t} \in \mathbb{T}_n$, $\mathbb{P}_\lambda^{(n)} = \underbrace{\mathbb{P}_{\tilde{\lambda}}(\,\cdot \mid |T| = n)}_{\text{the law of a } GW_{\tilde{\lambda}} \text{ tree conditioned on having } n \text{ vertices}}$

$\boxed{\text{By changing } c \text{, one adjusts the mean } \sum_{i \geq 0} i \tilde{\lambda}(i) \text{ of } \tilde{\lambda}}$

Remark 2 If $\sum_{n \geq 1} Z_n < \infty$, set $\overline{\mathbb{P}}_\lambda(\underline{t}) = \frac{w(\underline{t})}{\sum_{n \geq 1} Z_n}$.

Then $\forall n \geq 1$, $\overline{\mathbb{P}}_\lambda(\,\cdot \mid |T| = n) = \mathbb{P}_\lambda^{(n)}$.

This is used in simulation: if you sample according to $\overline{\mathbb{P}}_\lambda$ and only keep objects of size n, you get the law $\mathbb{P}_\lambda^{(n)}$!

4.) Coding by random walks.

Fix: $\underline{t} \in \Pi_f$ and let $u_0, u_1, \dots, u_{|t|-1}$ its vertices in lexicographical order. Denote by $(\widetilde{W}_i(\underline{t})\,;\, 0 \le i \le |\underline{t}|)$ its Lukasiewicz path defined by

$$\begin{cases} \widetilde{W}_0(\underline{t}) = 0 \\ \widetilde{W}_{i+1}(\underline{t}) = \widetilde{W}_i(\underline{t}) + (k_{u_i} - 1) \end{cases} \quad \text{for } 0 \le i \le |t|-1.$$

We know that

$$\Pi_f \longrightarrow \overline{S}^{(1)}$$

$$\underline{t} \longmapsto (\widetilde{W}_{i+1}(\underline{t}) - \widetilde{W}_i(\underline{t})\,;\, 0 \le i \le |\underline{t}|-$$

is a bijection.

Now let $(W_n)_{n \ge 0}$ be a random walk on $\mathbb{Z}$ such that $\mathbb{P}(W_1 = k) = \mu(k+1)$ for $k = -1, 0, 1, \dots$

Set $\zeta_1 = \inf\{n \ge 1,\ W_n = -1\}$.

Proposition Let T be a random tree with law $\mathbb{P}_\mu$. Then

$(\widetilde{W}_0(T), \widetilde{W}_1(T), \dots, \widetilde{W}_{|T|}(T))$ and $(0, W_1, \dots, W_{\zeta_1})$ have the same

Proof: Fix $n \geq 1$ and $x_1, \dots, x_n \geq -1$

Set $A = \mathbb{P}(\widetilde{W}_1(T) = x_1, \widetilde{W}_2(T) - \widetilde{W}_1(T) = x_2, \dots, \widetilde{W}_n(T) - \widetilde{W}_{n-1}(T) = x_n)$

$B = \mathbb{P}(W_1 = x_1, W_2 - W_1 = x_2, \dots, W_n - W_{n-1} = x_n)$

We show that $A = B$.

If $(x_1, \dots, x_n) \notin \overline{S}_n^{(1)}$, then $A = B = 0$.

If $(x_1, \dots, x_n) \in \overline{S}_n^{(1)}$, let $\underline{t}$ be the tree with Lukasiewicz path $(0, x_1, x_1 + x_2, \dots)$

then

- $A = \mathbb{P}(T = \underline{t}) = \prod_{u \in \underline{t}} \mu(k_u) = \prod_{i=1}^{n} \mu(x_i + 1)$

- $B = \mathbb{P}(W_1 = x_1, W_2 - W_1 = x_2, \dots, W_n - W_{n-1} = x_n, \zeta_1 = n)$
$= \mathbb{P}(W_1 = x_1, W_2 - W_1 = x_2, \dots, W_n - W_{n-1} = x_n)$
$= \prod_{i=1}^{n} \mu(x_i + 1)$.

this completes the proof ☒

Corollary 0
- Under $\mathbb{P}_\mu$, $|T|$ has the same law as ζ_1
- Under $\mathbb{P}_\mu(\cdot \mid |T| = n)$, $(\widetilde{W}_0(T), \dots, \widetilde{W}_n(T))$ has the same law as $(W_0, W_1, \dots, W_n)$ conditionally on $\zeta_1 = n$.

Notation: For $i \geq 1$, set $X_i = W_i - W_{i-1}$ and
$\overrightarrow{X}_n = (X_1, X_2, \dots, X_n)$.

Corollary 1 We have $\mathbb{P}_\mu(|T| = n) = \frac{1}{n} \mathbb{P}(W_n = -1)$.

Proof We have

$\mathbb{P}_\mu(|T| = n) = \mathbb{P}(\zeta_1 = n) = \mathbb{P}(\overrightarrow{X}_n \in \overline{S}_n^{(1)})$

$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(\overrightarrow{X}_n^{(i)} \in \overline{S}_n^{(1)})$ because $\overrightarrow{X}_n^{(i)} \overset{(d)}{=} \overrightarrow{X}_n$.

$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{1}_{\overrightarrow{X}_n \in S_n^{(1)}} \cdot \mathbb{1}_{i \in I_{\overrightarrow{X}_n}} \right]$

$= \frac{1}{n} \mathbb{E}\left[\mathbb{1}_{\overrightarrow{X}_n \in S_n^{(1)}} \left(\sum_{i=1}^{n} \mathbb{1}_{i \in I_{\overrightarrow{X}_n}} \right) \right] = \frac{1}{n} \mathbb{P}(\overrightarrow{X}_n \in S_n^{(1)})$

because $|I_{\vec{X}_n}|=1$ by the cyclic lemma.

But $\mathbb{P}(\vec{X}_n \in S_n^{(1)}) = \mathbb{P}(W_n=-1)$. $\blacksquare$

Corollary 2 Under $\mathbb{P}_\mu(\cdot \mid |T|=n)$, the law of $(\tilde{W}_1-\tilde{W}_0, \ldots, \tilde{W}_n-\tilde{W}_{n-1})$ is the same as the law of $\vec{X}_n^{(i_*)}$ under $\mathbb{P}(\cdot \mid W_n=-1)$, where i_* is the unique element of $I_{\vec{X}_n}$.
In addition, under $\mathbb{P}(\cdot \mid W_n=-1)$, $\vec{X}_n^{(i_*)} \perp\!\!\!\perp i_*$ and i_* is uniform on $\{1, 2, \ldots, n\}$.

Proof • We know that the law of $(\tilde{W}_1-\tilde{W}_0, \ldots, \tilde{W}_n-\tilde{W}_{n-1})$ under $\mathbb{P}_\mu(\cdot \mid |T|=n)$ is the same as the law of $\vec{X}_n$ under $\mathbb{P}(\cdot \mid \zeta_1=n)$ (Corollary 0)

Then for $\vec{x} \in S_n^{(1)}$, write

$$\underset{\mathbb{P}(\vec{X}_n=\vec{x},\,\zeta_1=n)}{\mathbb{P}(\vec{X}_n=\vec{x}, W_n=-1)} = \frac{1}{n}\sum_{i=1}^n \mathbb{P}(\vec{X}_n^{(i)}=\vec{x}, W_n=-1)$$

$$= \frac{1}{n}\mathbb{E}\Big[\mathbb{1}_{W_n=-1}\Big\{\sum_{i=1}^n \mathbb{1}_{\vec{X}_n^{(i)}=\vec{x}}\Big\}\Big]$$

$$= \frac{1}{n}\mathbb{E}\big[\mathbb{1}_{W_n=-1,\, \vec{X}_n^{(i_*)}=\vec{x}}\big]$$

But $\mathbb{P}(\zeta_1=n) = \frac{1}{n}\mathbb{P}(W_n=-1)$.

Hence

$$\mathbb{P}(\vec{X}_n=\vec{x} \mid \zeta_1=n) = \mathbb{P}(\vec{X}_n^{(i_*)}=\vec{x} \mid W_n=-1).$$

• Write

$$\mathbb{P}(i_*=k, \vec{X}_n^{(i_*)}=\vec{x}, W_n=-1) = \mathbb{P}(i_*=k, \vec{X}_n^{(i_*)}=\vec{x}) \qquad (1\le k\le n,\ \vec{x}\in S_n^{(1)})$$

$$= \mathbb{P}(i_*=k, \vec{X}_n^{(k)}=\vec{x})$$

$$= \mathbb{P}(\vec{X}_n^{(k)}=\vec{x}) \quad \text{since } |I_{\vec{x}_n}|=1$$

$$= \mathbb{P}(\vec{X}_n^{(k)}=\vec{x}, W_n=-1)$$

$$= \frac{1}{n}\mathbb{P}(W_n=-1, \vec{X}_n^{(i_*)}=\vec{x})$$

$$= \frac{1}{n}\mathbb{P}(\vec{X}_n^{(i_*)}=\vec{x}) = \frac{1}{n}\mathbb{P}(\vec{X}_n^{(i_*)}=\vec{x}, W_n=-1)$$

Divide everything by $\mathbb{P}(W_n=-1)$ to get $\mathbb{P}(i_*=k, \vec{X}_n^{(i_*)}=\vec{x} \mid W_n=-1) = \frac{1}{n}\mathbb{P}(\vec{X}_n^{(i_*)}=\vec{x} \mid W_n=-1)$

Extension to forests

Denote by $\mathbb{P}_{\mu,j}$ the probability measure on $\mathbb{T}_f^j$ which is the law of the forest of j independent GW_μ trees.
Denote by $(\widehat{W}_0(f), \ldots, \widehat{W}_{|f|}(f))$ its Lukasiewicz path.

Then under $\mathbb{P}_{\mu,j}$,

$$\boxed{(\widehat{W}_0(f), \ldots, \widehat{W}_{|f|}(f)) \overset{(d)}{=} (W_0, W_1, \ldots, W_{\zeta_k})}$$

where $\zeta_k = \inf\{k \geq 1 ; W_k = -1\}$.

In addition,

$$\boxed{\mathbb{P}_{\mu,j}(|F| = n) = \frac{j}{n} \mathbb{P}(W_n = -1)}$$

We would like therefore to estimate $\mathbb{P}(W_n = -1)$!.