LECTURE 3

Proof of the cyclic lemma

The main idea is the following claim:

<u>Claim</u> $|I_{\underline{x}}|$ does not change when $(a, \overbrace{-1, \dots, -1}^{a \text{ times}})$ is concatenated to the left of $\underline{x}$

<u>Proof of the claim</u>

Set $\tilde{\underline{x}} = (a, \overbrace{-1, \dots, -1}^{a \text{ times}}, x_1, \dots, x_n)$ where $a \geq 1$ and $\underline{x} = (x_1, \dots, x_n)$

- first, it is clear that $0 \in I_{\tilde{\underline{x}}} \iff 0 \in I_{\underline{x}}$.
- if $0 < i \leq n-1$, we have

$$\tilde{\underline{x}}^{(i+a+1)} = (x_{i+1}, \dots, x_n, a, -1, \dots, -1, x_1, \dots, x_i)$$

From this, it is easy to see that

$$i \in I_{\underline{x}} \iff i+a+1 \in I_{\tilde{\underline{x}}}$$

- If $0 < i \leq a+1$, we check that $i \notin I_{\tilde{\underline{x}}}$:

$$\tilde{\underline{x}}^{(i)} = (\underbrace{-1, -1, \dots, -1}_{a-i+1 \text{ times}}, x_1, x_2, \dots, x_n, a, -1, \dots, -1)$$

the sum of all the elements of $\tilde{\underline{x}}^{(i)}$ up to the element x_n is $x_1 + \dots + x_n - (a-i+1) = -k - (a-i+1)$
$< -k$.

Hence $\tilde{\underline{x}}^{(i)} \notin I_{\tilde{\underline{x}}}$.

this proves the claim. $\boxtimes$

We now prove the cyclic lemma by induction on n.

- if $n=k$: OK ($\bar{x} = (-1, \dots, -1)$)
- Assume that the result is true for $j = k, \dots, n-1$. with $n > k$.

Take $\bar{x} \in \bar{S}_n^{(k)}$. Since $\bar{x}$ has a nonnegative element, we can, up to cyclic permutation, assume that $x_1 \geq 0$.

Let $1 = i_1 < i_2 < \cdots < i_m$ be the indices i for which $x_i \geq 0$.
Set $i_{m+1} = n+1$ by convention.
Hence

$$-k = \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} \Big(x_{i_j} - \underbrace{(i_{j+1} - i_j - 1)}^{\text{number of } -1 \text{ following } x_{i_j}} \Big)$$

Since this sum is negative, there exists j s.t.

$$x_{i_j} \leq i_{j+1} - i_j - 1$$

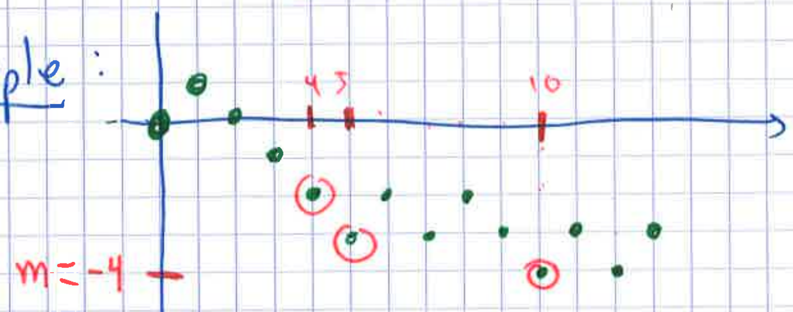
$\Rightarrow$ x_{i_j} is followed by at least x_{i_j} times "-1"

Thus $I_{\underline{x}} = I_{\underline{\tilde{x}}}$ where $\underline{\tilde{x}}$ is obtained from $\underline{x}$ by removing x_{i_j} followed by x_{i_j} times "-1".
We conclude by the induction hypothesis. $\boxtimes$

Corollary For $\underline{x} \in S_n^{(k)}$, set $m = \inf\{x_1 + \cdots + x_i \,;\, 1 \leq i \leq n\}$
and $\zeta_i(\underline{x}) = \inf\{ j \geq 1 \,;\, x_1 + \cdots + x_j = m+i-1\}$
Then $I_{\underline{x}} = \{ \zeta_1(\underline{x}), \zeta_2(\underline{x}), \ldots, \zeta_k(\underline{x}) \}$

Indeed, this property is invariant under concatenation of $(a, -1, \ldots, -1)$!

Example:



$\in S_{13}^{(3)}$

$I_{\underline{x}} = \{4, 5, 10\}$

III Generating functions

1) Examples

Trees Set $T(x) = \sum_{n \geq 0} |\mathbb{T}_n| \cdot x^n$.

$T(x)$ is a formal power series: $\mathbb{R}[[X]]$ is the set of real-valued sequences indexed by $\mathbb{Z}_+$, with usual addition and with convolution products (i.e like polynomials with infinitely many coefficients).

Note that $T(x) = \sum_{\underline{t} \in \mathbb{T}_f} x^{|\underline{t}|}$: T is called generating function of trees with weight x by vertex

Claim: $T(x) = \dfrac{x}{1 - T(x)}$

proof: tree = root + subtrees. Hence

$$T(x) = \sum_{\underline{t} \in \mathbb{T}_f} \sum_{k \geq 0} \mathbb{1}_{k_\emptyset(\underline{t}) = k} \, x^{|\underline{t}|}$$

$$= \sum_{k \geq 0} \sum_{\underline{t}_1, \ldots, \underline{t}_k} x^{1 + |\underline{t}_1| + \cdots + |\underline{t}_k|} = \sum_{k \geq 0} x \cdot (T(x))^k$$

$$= x \sum_{k \geq 0} (T(x))^k.$$

But $\sum_{k \geq 0} T(x)^k = \dfrac{1}{1 - T(x)}$ $\left(\text{indeed, } (1 - T(x))\left(\sum_{k \geq 0} T(x)^k\right) = 1 \text{ in } \mathbb{R}[[X]]\right)$

Corollary: $T(x) = \dfrac{1 - \sqrt{1 - 4x}}{2}$

indeed, $T(x) - T(x)^2 = x$.

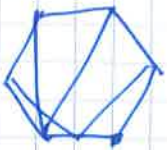
NB: we do not enter details on limit operations in $\mathbb{R}[[X]]$, see Combinatorial Enumeration (Chap 1) by Goulden & Jackson for details

NB $T(x) = x\,\phi(T(x))$ with $\phi(x) = \dfrac{1}{1 - x}$.

Dissections Set $P_n = \left\{ e^{\frac{2i\pi k}{n}} ; 0 \leq k \leq n-1 \right\}$.

A dissection of P_n is a collection of noncrossing diagonals

Example:



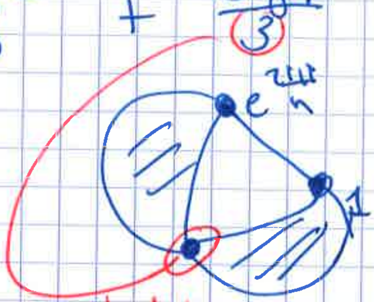
Set $\mathcal{D}_n = \{ \text{dissections of } P_n \}$

and $D(z) = \sum_{n \geq 2} |\mathcal{D}_n| z^n \quad \left(= \sum_{D \text{ dissection}} z^{\text{size}(D)} \right)$

By convention, we set $\mathcal{D}_2 = \{ | \}$. Then $\mathcal{D}_3 = \{ \Delta \}$, $\mathcal{D}_4 = \{ \diamond, \diamond, \diamond \}$

Decompose a dissection according to the number r of edges of the face adjacent to $[1, e^{\frac{2i\pi}{n}}]$

$$D(z) = \overbrace{z^2}^{r=2} + \overbrace{\frac{D(z)^2}{z^{(3)}}}^{r=3} + \overbrace{\frac{D(z)^3}{z^{(2)}}}^{r=4} + \cdots$$



counted twice

counted twice

Thus, if $E(z) = \frac{D(z)}{z}$, we get

$$E(z) = z + E(z)^2 + E(z)^3 + \cdots = z + \frac{E(z)^2}{1 - E(z)}$$

$$\Rightarrow z = E(z) \cdot \frac{1 - 2E(z)}{1 - E(z)}$$

Hence $E(z) = z\,\phi(E(z))$ with $\phi(z) = \frac{1-z}{1-2z}$

and $E(z) = \frac{1 + z + \sqrt{1 - 6z + z^2}}{4}$

2) Lagrange inversion theorem.

If $\phi(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{R}[[x]]$, set $[x^n]\phi = a_n$.

Theorem Let $\phi \in \mathbb{R}[[X]]$ such that $\phi(0) \neq 0$

1) there exists a unique $w \in \mathbb{R}[[X]]$ such that $w(0) = 0$ and $w(z) = z\,\phi(w(z))$ $(\star)$

2) For every $g \in \mathbb{R}[[X]]$,
$$[\ ^n]\, g(w(z)) = \frac{1}{n}[z^{n-1}]\, g'(z)\,\phi(z)^n$$

N.B 2) $\Rightarrow$ in particular, $[z^n]\, w(z)^k = \frac{1}{n}[z^{n-1}]\, k\, z^{k-1}\phi(z)^n$
$$= \frac{k}{n}[z^{n-k}]\,\phi(z)^n.$$

Proof 1) Not proved here (Idea: $(\star)$ determines the coefficients of w in a unique way)
See Combinatorial Enumeration of Goulden & Jackson Chap 1, for more on formal power series.

2) We exhibit the solution and check that it works.
Set $\phi(z) = \sum_{i\geq 0} a_i z^i$ and set
$$w(z) = \sum_{t \in \Pi_g} z^{|t|} \prod_{u \in t} a_{k_u}$$
Then
$$w(z) = \sum_{k\geq 0} \sum_{t\in\Pi_g} 1_{k_\phi(t)=k}\, z^{|t|} \prod_{u\in t} a_{k_u}$$
$$= \sum_{k\geq 0} \sum_{t_1,\dots,t_k} (a_k z) \cdot \prod_{i=1}^{k} z^{|t_i|} \prod_{u\in |t_i|} a_{k_u}$$
$$= \sum_{k\geq 0} a_k z\, w(z)^k = z\,\phi(w(z)).$$

Next time, we will check that indeed, for this w,
$$[z^n]\, w(z)^k = \frac{k}{n}[z^{n-k}]\,\phi(z)^n$$