II Plane trees and the cyclic lemma

1) Definitions

Goal: define plane rooted trees: 

- Set $\mathbb{N} = \{1, 2, \dots\}$, $\mathcal{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ $\quad (\mathbb{N}^0 = \{\emptyset\})$
- If $u = u_1 \dots u_n \in \mathcal{U}$, let $|u| = n$ be the generation of u
- If $u = u_1 \dots u_n$ and $v = v_1 \dots v_m \in \mathcal{U}$, set

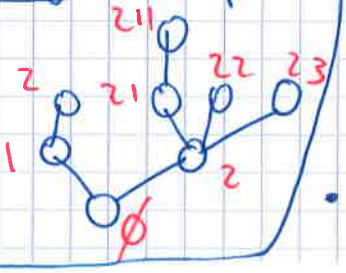
$$uv = u_1 \dots u_n v_1 \dots v_m.$$

Def Let Π be the set of all subsets $\underline{t} \subset \mathcal{U}$ s.t.:

1) $\emptyset \in \underline{t}$
2) if $ui \in \underline{t}$ $(i \geq 1)$, then $u \in \underline{t}$
3) For every $u \in \underline{t}$, there exists $k_u(\underline{t}) \geq 0$ s.t.

$$\forall j \geq 1, \quad uj \in \underline{t} \iff j \leq k_u(\underline{t}).$$

Π is the set of all (plane rooted) trees.

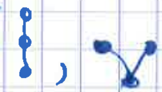
Ex: 

- $k_u(\underline{t})$: number of children of u
- $|\underline{t}|$ is the size of the tree
- We equip $\mathcal{U}$ with the lexicographical order:

$u \leq v$ if we have $v = uw$ or if we can write $\begin{cases} u = u_0 i u' \\ v = u_0 j v' \end{cases}$ with $i < j$.

We set $\Pi_f = \{\underline{t} \in \Pi ;\ \underline{t} \text{ finite}\}$ and $\Pi_n = \{\underline{t} \in \Pi ;\ |\underline{t}| = n\}$

2) The Lukasiewicz path of a tree

$\Pi_3 = \{$  $\}$

For $k, n \geq 1$, set

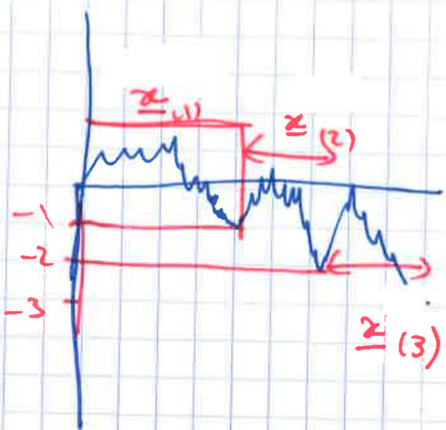
$$S_n^{(k)} = \{(x_1, \dots, x_n) \in \{-1, 0, 1, \dots\}^n ;\ x_1 + \dots + x_n = -k\}$$

$$\overline{S}_n^{(k)} = \{(x_1, \dots, x_n) \in S_n^{(k)} ;\ x_1 + \dots + x_j > -k \text{ for } 1 \leq j \leq n-1\}$$

$$S^{(k)} = \bigcup_{n \geq 1} S_n^{(k)} \qquad \overline{S}^{(k)} = \bigcup_{n \geq 1} \overline{S}_n^{(k)}$$

If $\underline{x} = (x_1, \dots, x_n) \in \overline{S}_n^{(k)}$ and $\underline{y} = (y_1, \dots, y_m) \in \overline{S}_m^{(k')}$, we write $\underline{x}\,\underline{y} = (x_1, \dots, x_n, y_1, \dots, y_m)$ for the concatenation of $\underline{x}$ and $\underline{y}$. In particular, $\underline{x}\,\underline{y} \in \overline{S}_{n+m}^{(k+k')}$.

Remark: If $\underline{x} \in \overline{S}^{(k)}$, we can write $\underline{x} = \underline{x}_{(1)} \cdots \underline{x}_{(k)}$ with $\underline{x}_{(i)} \in \overline{S}^{(1)}$ for every $1 \leq i \leq k$ in a unique way:



Formally, for $1 \leq i \leq k$, set

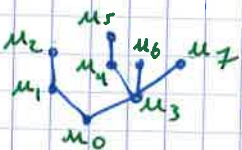
$$\zeta_i = \inf\{ j \geq 1;\ x_1 + \dots + x_j = -i \}$$

Then
$$\underline{x}_{(1)} = (x_1, x_2, \dots, x_{\zeta_1})$$
$$\underline{x}_{(2)} = (x_{\zeta_1+1}, \dots, x_{\zeta_2})$$
$$\vdots$$
$$\underline{x}_{(k)} = (x_{\zeta_{k-1}+1}, \dots, x_{\zeta_k})$$

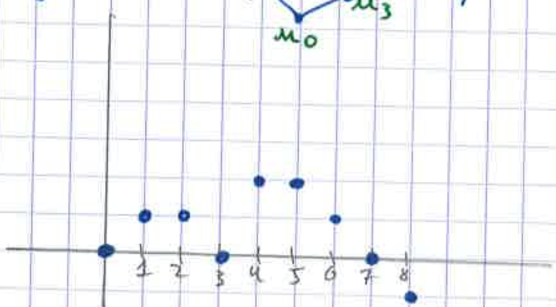
For $\underline{t} \in \mathbb{T}_f$, let $u_0, u_1, \dots, u_{|t|-1}$ be its vertices in lexicographical order.

Prop. For $n \geq 1$,
$$\Phi_n : \mathbb{T}_n \longrightarrow \overline{S}_n^{(1)}$$
$$\underline{t} \longmapsto (k_{u_{i-1}} - 1\ ;\ 1 \leq i \leq n)$$
is a bijection.

The sequence $\left(\sum_{i=1}^{j} (k_{u_{i-1}} - 1)\ ;\ 0 \leq j \leq |t| \right)$ is called the Lukasiewicz path of $\underline{t}$

Example if $\underline{t} =$ 

, its Lukasiewicz path is:



Proof Take $\underline{t} \in \mathbb{T}_n$. We first check that $\Phi_n(\underline{t}) \in \overline{S}_n^{(1)}$.

For $1 \leq j \leq n$, we have:

$$\sum_{i=1}^{j} (k_{u_{i-1}} - 1) = \underbrace{\sum_{i=1}^{j} k_{u_{i-1}}}_{\text{counts the number of children of } u_0, u_1, \dots, u_{j-1}} - j \quad (*)$$

If $j < n$, $u_1, \dots, u_j$ are among the children of $u_0, u_1, \dots, u_{j-1}$ so that $(*)$ is ≥ 0

If $j = n$, $\sum_{i=1}^{n} k_{u_{i-1}}$ counts all the vertices who are children of someone, which is everyone except the root.
Hence $(*) = -1$.

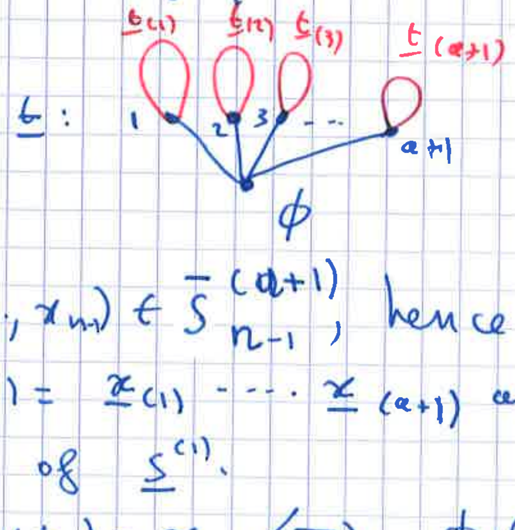
$\boxed{\text{Thus } \phi_n(\underline{t}) \in \bar{S}_n^{(1)}.}$

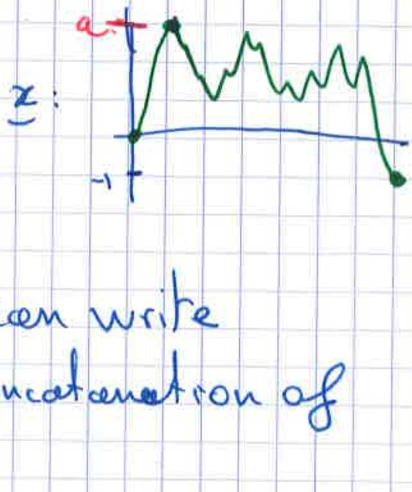
Next, we check that Φ_n is a bijection by induction.

- for $n=1$, ok.
- Φ_j is bijective for $j = 1, \dots, n-1$. We show that Φ_n is bijective

Fix $\underline{x} = (a, x_1, \dots, x_{n-1}) \in \bar{S}_n^{(1)}$.

If $\Phi_n(\underline{t}) = \underline{x}$, then $k_\phi(\underline{t}) = a+1$, and $(x_1, \dots, x_n)$ must be the concatenation of the images by Φ of the subtrees $\underline{t}_{(1)}, \dots, \underline{t}_{(a+1)}$ attached to the children of ϕ:





But $(x_1, \dots, x_{n-1}) \in \bar{S}_{n-1}^{(a+1)}$, hence we can write $(x_1, \dots, x_{n-1}) = \underline{x}_{(1)} \cdots \underline{x}_{(a+1)}$ as the concatenation of $a+1$ elements of $\underline{S}^{(1)}$.

Hence $\Phi_n(\underline{t}) = \underline{x} \iff \phi_{|\underline{t}_{(i)}|}(\underline{t}_{(i)}) = \underline{x}_{(i)} \quad \forall\, 1 \leq i \leq a+1$

$$\iff \underline{t} = \{\phi\} \cup \bigcup_{i=1}^{a+1} i\, \Phi_{|\underline{t}_{(i)}|}^{-1}(\underline{x}_{(i)})$$

by induction hypothesis since $|\underline{t}_{(i)}| < |\underline{t}|$. ☒

Corollary $|\Pi_n| = |\overline{S}_n^{(1)}|$.

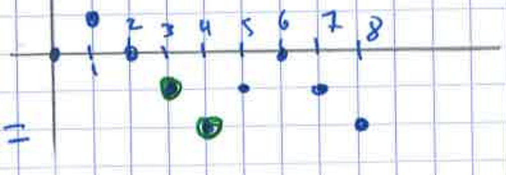
Pb How to count $|\overline{S}_n^{(k)}|$?

3) The cyclic lemma.

We identify $i \in \mathbb{Z}/n\mathbb{Z}$ with its representative in $\{0, 1, \dots, n-1\}$.
For $\overline{x} \in S_n^{(k)}$ and $i \in \mathbb{Z}/n\mathbb{Z}$, set $\overline{x}^{(i)} = (x_{i+1}, \dots, x_{i+n})$
where the addition is modulo n.

Definition: For $\overline{x} \in S_n^{(k)}$, set $I_{\overline{x}} = \{ i \in \mathbb{Z}/n\mathbb{Z} \,;\, \overline{x}^{(i)} \in \overline{S}_n^{(k)} \}$

Example

$\overline{x} =$ 

$I_{\overline{x}} = \{3, 4\}$

Proposition [Cyclic Lemma]

For every $\overline{x} \in S_n^{(k)}$, $|I_{\overline{x}}| = k$.

Before proving this lemma, let's see some applications.

Set

$$\chi : \bar{S}_n^{(k)} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \{ (\underline{x}, i) \, ; \, \underline{x} \in S_n^{(k)} \text{ and } \underline{x}^{(i)} \in \bar{S}_n^{(k)} \}$$

$$(\underline{x}, i) \longmapsto (\underline{x}^{(i)}, -i).$$

Note that χ is well defined, since if $\underline{x} \in \bar{S}_n^{(k)}$ and $i \in \mathbb{Z}/n\mathbb{Z}$, we have $(\underline{x}^{(i)})^{(-i)} = \underline{x} \in \bar{S}_n^{(k)}$.

Claim: χ is a bijection

Indeed, χ is injective: if $\chi(\underline{x}_1, i_1) = \chi(\underline{x}_2, i_2)$, then $-i_1 = -i_2$, so $i_1 = i_2$, and since $\underline{x}_1^{(-i_1)} = \underline{x}_2^{(-i_2)}$, we get that $x_1 = x_2$

Also, χ is a surjection: if $\underline{x} \in S_n^{(k)}$ and $i \in I_{\underline{x}}$, then $\chi(\underline{x}^{(i)}, -i) = (\underline{x}, i)$.

Remark: the fact that χ is a bijection also follows from the fact that $\chi(\chi(\underline{x}, i)) = (\underline{x}, i)$ $\forall \, \underline{x} \in \bar{S}_n^{(k)}, i \in \mathbb{Z}/n\mathbb{Z}$!

Corollary: $|\bar{S}_n^{(k)}| = \frac{k}{n} |S_n^{(k)}|$.

Proof: since χ is a bijection,

$$|\bar{S}_n^{(k)}| \times n = |\bar{S}_n^{(k)} \times \mathbb{Z}/n\mathbb{Z}| = |\{ (\underline{x}, i) ; \underline{x} \in S_n^{(k)} \text{ and } i \in I_{\underline{x}} \}|$$

$$= |S_n^{(k)}| \times k \quad \text{by the cyclic lemma}$$

$\blacksquare$

Lemma: We have $|S_n^{(k)}| = \binom{2n-k-1}{n-1}$

Proof:

$$|S_n^{(k)}| = |\{ (x_1, .., x_n) \in \{-1, 0, 1, ..\}^n ; x_1 + \cdots + x_n = -k \}|$$

$$= |\{ (x_1, .., x_n) \in \{1, 2, 3, ..,\}^n ; x_1 + \cdots + x_n = 2n - k \}|$$

$$= |\{ 0 < y_1 < \cdots < y_{n-1} < 2n-k \}|$$

(by setting $y_i = x_1 + \cdots + x_i$

$$= \binom{2n-k-1}{n-1}$$

$\blacksquare$

Corollary $|\Pi_n| = |\bar{S}_n^{(1)}| = \frac{1}{n} \cdot \binom{2n-2}{n-1}$!