Solution to the exercise

First solution: By the strong law of large numbers,

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s} \mathbb{E}[Z_1].$$

In particular, a.s, there exists N (which is random) such that

$$n \geq N \Rightarrow \left| \frac{S_n}{n} - \mathbb{E}[Z_1] \right| \leq \frac{|\mathbb{E}[Z_1]|}{2}$$

Hence $n \geq N \Rightarrow S_n \neq 0$.

Hence a.s $\sum_{n=0}^{\infty} \mathbb{1}_{\{S_n=0\}} < \infty$, so that 0 is transient.

$\boxtimes$

Second solution: Set $\phi(t) = \mathbb{E}[e^{itZ_1}]$.

We check that $\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1-\phi(t)}\right) dt < \infty$ and apply Chung-Fuchs theorem.

It is enough to check that $\int_0 \mathrm{Re}\left(\frac{1}{1-\phi(t)}\right) dt < \infty$ (exercise)

By Taylor's formula, $\phi(t) = 1 + i\,\mathbb{E}[Z_1]t + o(t)$

Hence $\frac{1}{1-\phi(t)} = \frac{1}{-i\mathbb{E}[Z_1]t} + \frac{1}{-i\mathbb{E}[Z_1]t} o(1)$

Hence $\mathrm{Re}\left(\frac{1}{1-\phi(t)}\right) \xrightarrow[t\to 0]{} 0$

Hence $\int_0 \mathrm{Re}\left(\frac{1}{1-\phi(t)}\right) dt < \infty$.

$\boxtimes$