Exercise given at the end of Lecture 7.

Let μ be a critical offspring distribution on $\mathbb{Z}_+$, with finite and positive variance σ^2. Recall that for every tree τ and $k \geq 1$, $[\tau]_k = \{u \in \tau; |u| \leq k\}$ is the tree τ cut above generation k, and that $\mathcal{T}_\infty$ denotes the μ-Galton–Watson tree conditioned to survived.

1. Let $\mathcal{T}_n$ be a Galton–Watson tree with offspring distribution μ, conditioned on having n *leaves*. Show that $\mathcal{T}_n$ converges locally in distribution to $\mathcal{T}_\infty$ as $n \to \infty$, that is for every finite tree τ and $k \geq 0$,
$$\mathbb{P}([\mathcal{T}_n]_k = \tau) \underset{n\to\infty}{\longrightarrow} \mathbb{P}([\mathcal{T}_\infty]_k = \tau)$$

2. Let $\mathcal{D}_n$ be a random dissection of $\{1, e^{2i\pi/n}, \dots, e^{2i(n-1)\pi/n}\}$, chosen uniformly at random among all dissections of $\{1, e^{2i\pi/n}, \dots, e^{2i(n-1)\pi/n}\}$.

   (i) Let $\mathcal{F}_n$ denote the degree of the face adjacent to the side $[1, e^{2i\pi/n}]$ of $\mathcal{D}_n$. Show that
$$\mathbb{P}(\mathcal{F}_n = k) \underset{n\to\infty}{\longrightarrow} (k-1)\left(\frac{2-\sqrt{2}}{2}\right)^{k-2}, \qquad k \geq 3.$$

   (ii) Let $\mathcal{V}_n$ denote the number of diagonals adjacent to the vertex corresponding to the complex number 1 in $\mathcal{D}_n$. Show that
$$\mathbb{P}(\mathcal{V}_n = k) \underset{n\to\infty}{\longrightarrow} (k+1)\cdot(2-\sqrt{2})^2(\sqrt{2}-1)^k, \qquad k \geq 0.$$