Solution of the exercise

First recall the following simple lemma:

Lemma Let A be a finite subset of a set B and let μ be a measure on B. Then μ is the uniform probability measure on A iff

1) $\mu(B \setminus A) = 0$
2) $\mu(\{a\}) = \mu(\{a'\}) \quad \forall a, a' \in A.$

Now fix $c > 0$ and define $\mu(0) = \frac{1-2c}{1-c}$, $\mu(1) = 0$, $\mu(i) = c^{i-1}$ $(i \geq 2)$

Let τ_n be a random tree with law $\mathbb{P}_\mu(\cdot \mid \lambda(\tau) = n)$.
We check that τ_n is uniform on L_n.

1) if $\underline{t} \notin L_n$, $\mathbb{P}(\tau_n = \underline{t}) = 0$

2) if $\underline{t} \in L_n$,

$$\mathbb{P}(\tau_n = \underline{t}) = \prod_{u \in \underline{t}} \mu(k_u)$$

$$= \mu(0)^n \cdot \prod_{\substack{u \in \underline{t} \\ u \text{ not} \\ \text{a leaf}}} c^{k_u - 1}$$

$$= \mu(0)^n \, c^{\sum_{\substack{u \in t \\ u \text{ not leaf}}} k_u - \sum_{\substack{u \in \underline{t} \\ u \text{ not} \\ \text{a leaf}}} 1}$$

$$= \mu(0)^n \, c^{(|\underline{t}|-1) - (|\underline{t}|-n)}$$

$$= \mu(0)^n \, c^{n-1},$$ which does not depend on $\underline{t}$!

This is useful, since dissections of P_n are in bijection with L_{n-1}!