Exercise: Find the generating function of (plane rooted) binary trees with n vertices.

Solution: A binary tree always has an odd number of vertices; and it has $2n+1$ vertices iff it has n leaves.

$\Rightarrow$ Denote by B_n the number of binary trees with n leaves.

and $B(z) = \sum_{n \geq 1} B_n z^n = \sum_{t \text{ binary}} z^{\text{number of leaves of } t}$

But a binary tree is either just the root, or the root with two binary trees grafted on top of it.

Hence $B(z) = z + B(z)^2$

Hence $B(z) = z\,\phi(B(z))$ with $\phi(x) = \frac{1}{1-x}$.

By Lagrange inv. theorem (uniqueness part),

$$B(z) = T(z)$$

$\uparrow$ gen. function of trees with n vertices.

$$\Rightarrow B_n = T_n = \frac{1}{n}\binom{2n-2}{n-1}$$

Hence the number of (plane rooted) binary trees with $2n+1$ vertices is $\frac{1}{n}\binom{2n-2}{n-1}$

NB: As we saw in the lecture, there is a nice bijection between B_n and T_n!