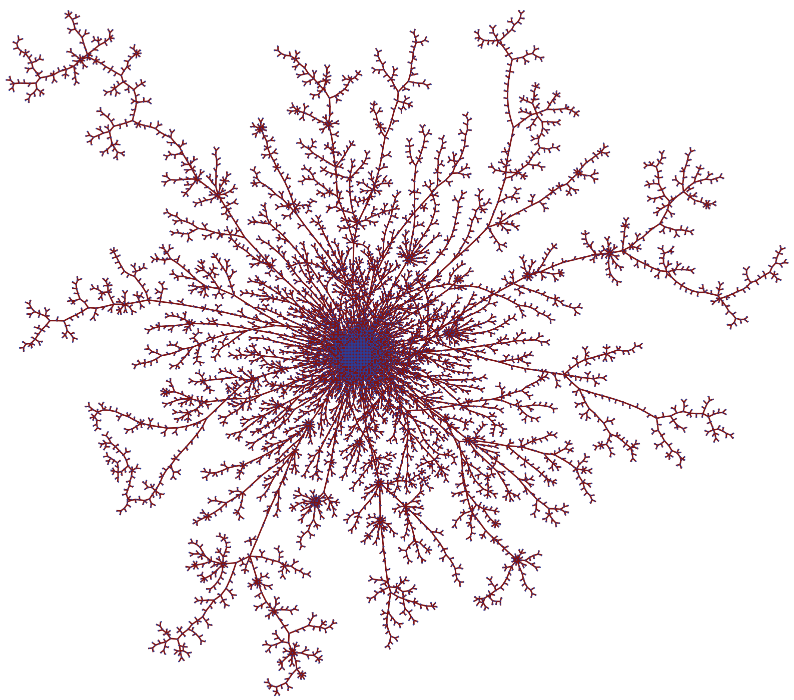
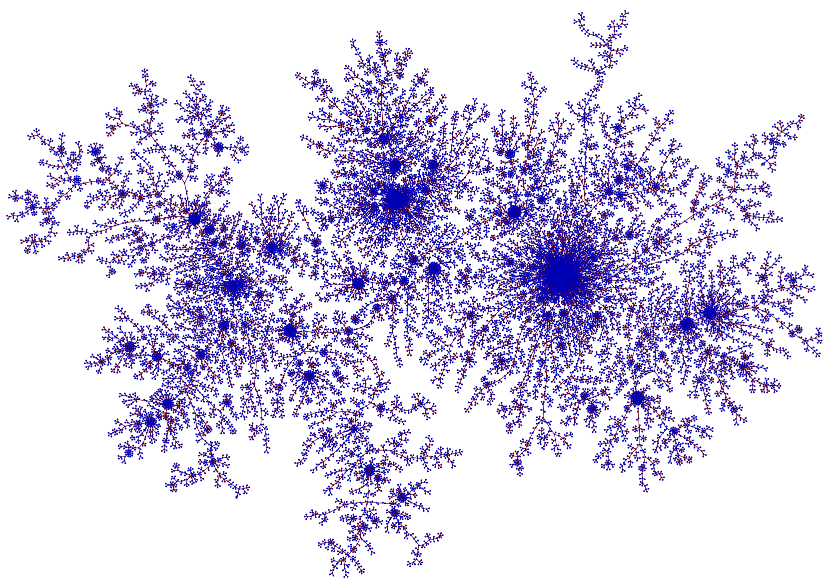
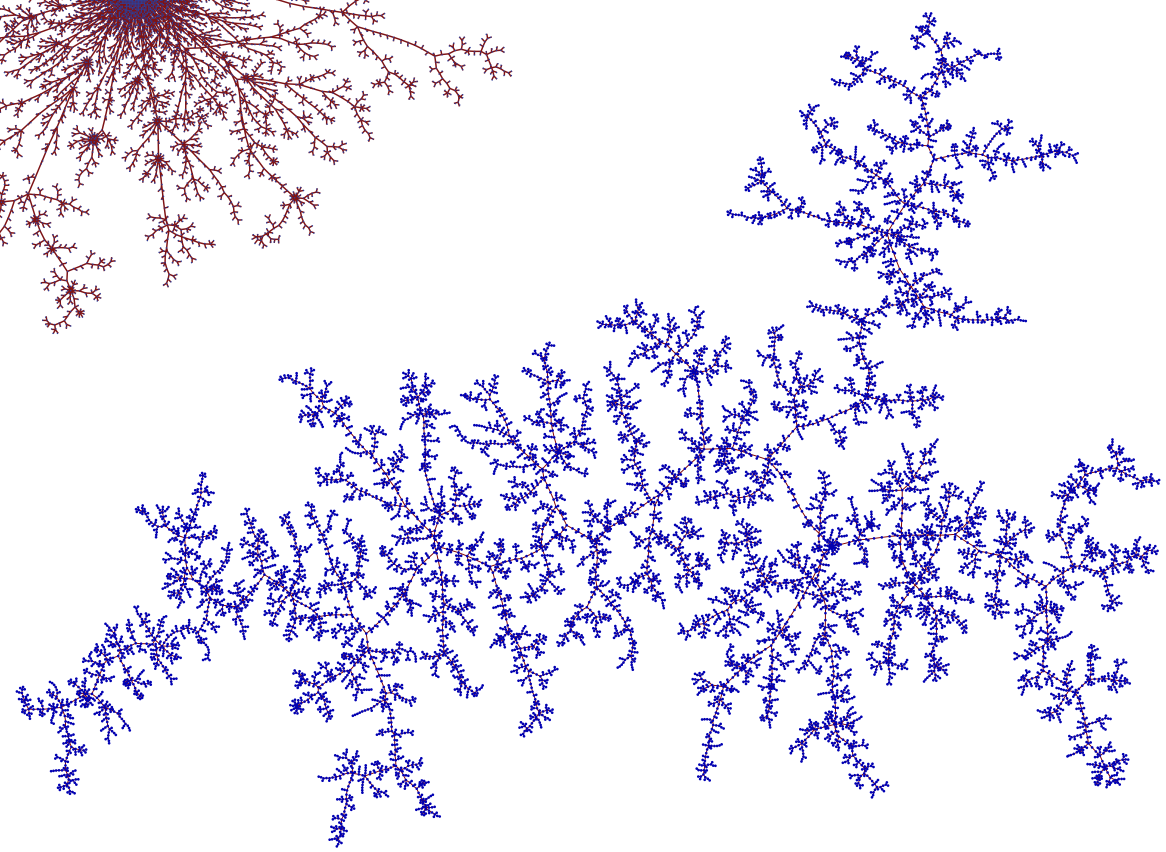
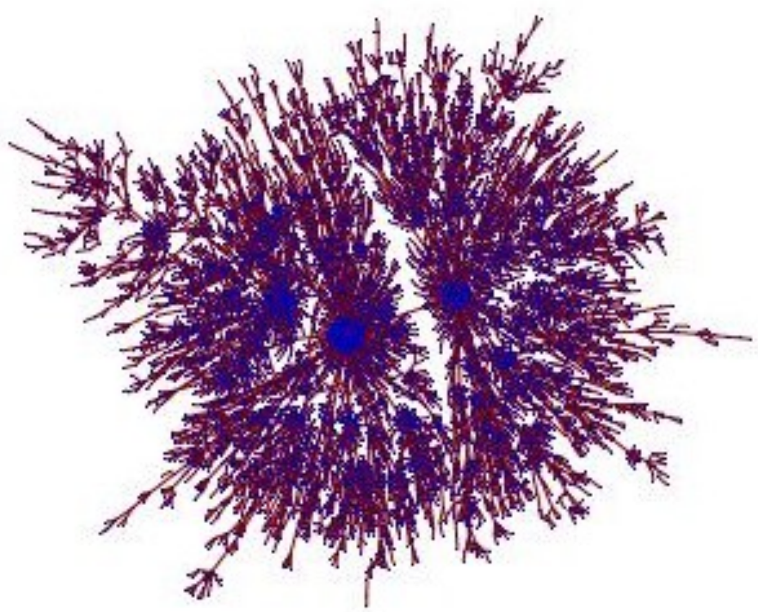
Phénomènes de condensation dans les arbres aléatoires









Igor Kortchemski

CNRS & École polytechnique

Masterclass M2 Géométrie | Probabilités, Nancy 2023

Motivation for studying limits

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Goal: study $\mathcal{X}_n$.

- Find the cardinal of $\mathcal{X}_n$. (bijective methods, generating functions)
- Understand the typical properties of $\mathcal{X}_n$. Let X_n be an element of $\mathcal{X}_n$ chosen *uniformly at random.* What can be said of X_n?
- A possibility to study X_n is to find a limiting object X such that $X_n \to X$ as $n \to \infty$.

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Let $(X_n)_{n \geqslant 1}$ be "discrete" objects converging towards a "limiting" object X:

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.
- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

Motivation for studying limits

Let $(X_n)_{n \geqslant 1}$ be "discrete" objects converging towards a "continuous" object X:

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Let $(X_n)_{n \geqslant 1}$ be "discrete" objects converging towards a "continuous" object X:

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- *What is the sense of the convergence when the objects are random?*

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$$X_n \underset{n \to \infty}{\longrightarrow} X.$$

- *In what space do the objects live?* Here, a metric space (Z, d)
- *What is the sense of the convergence when the objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left[F(X)\right]$$

for every continous bounded function $F : Z \to \mathbb{R}$.

Outline

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Stack triangulations (Albenque, Marckert)

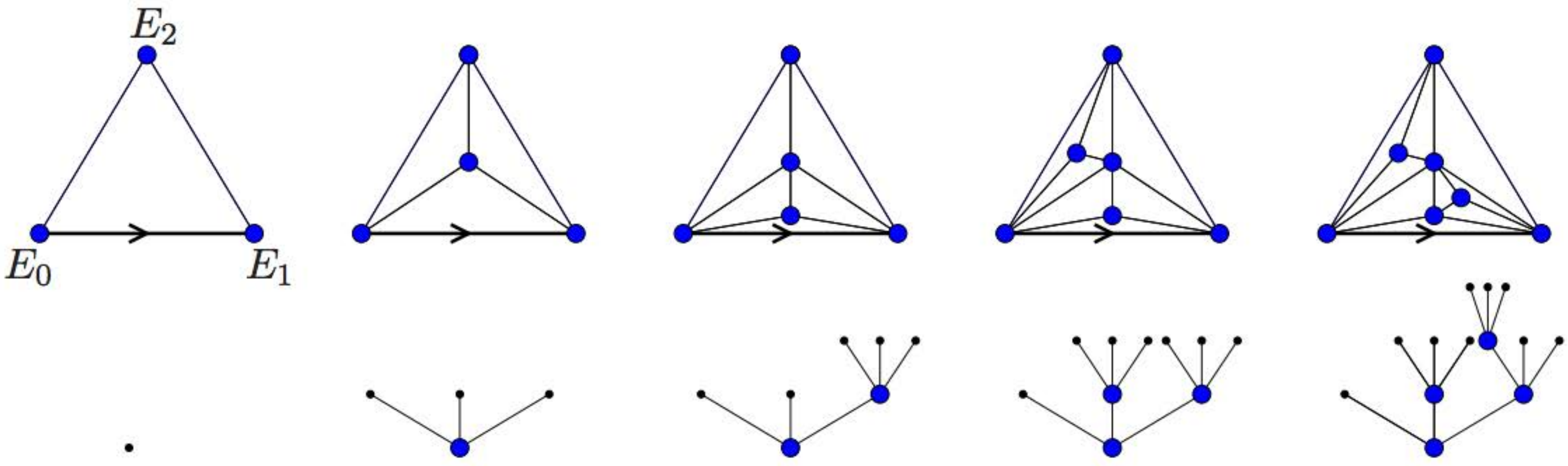


Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

Dissections (Curien, K.)

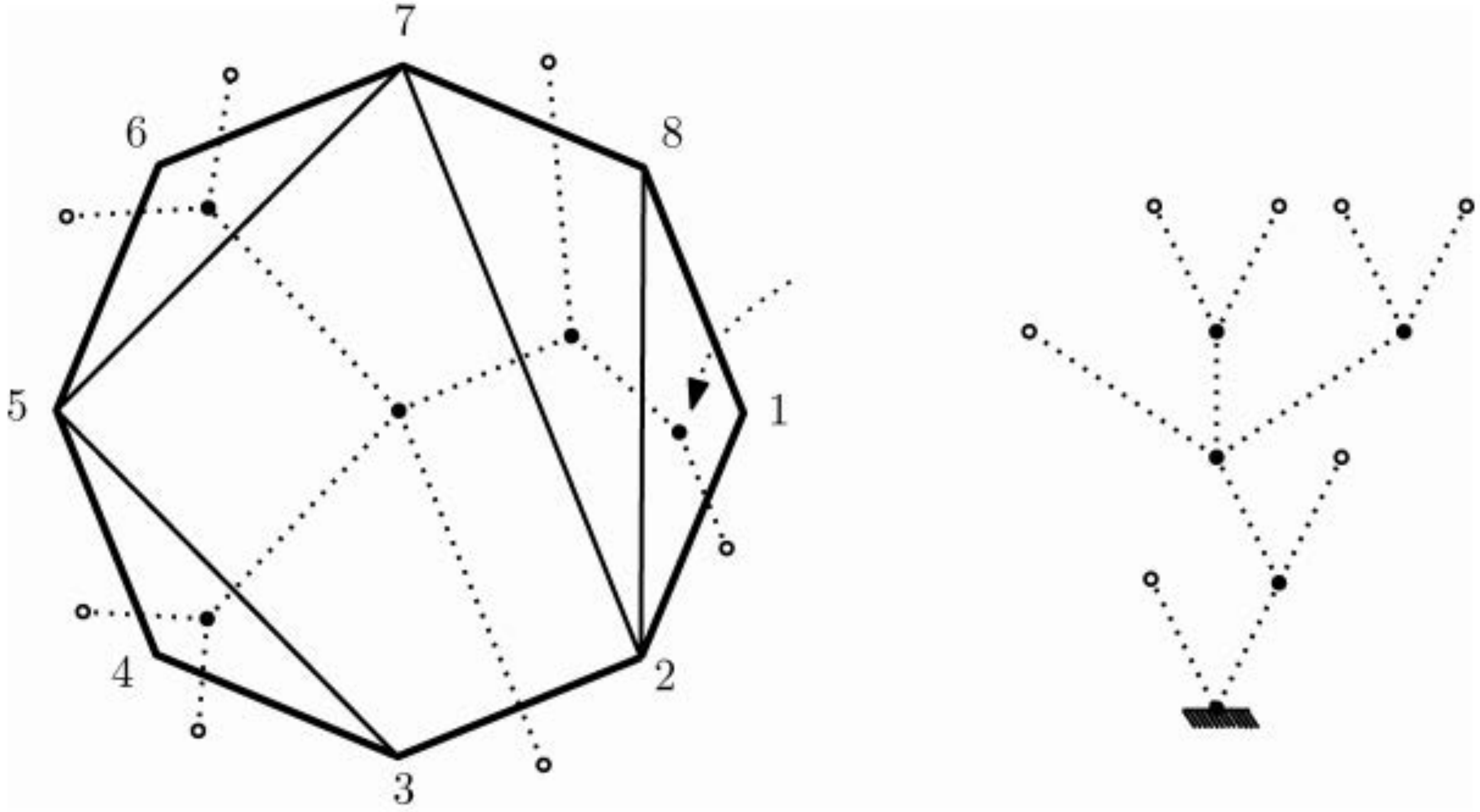


Fig. 4. The dual tree of a dissection of P_8, note that the tree has 7 leaves.

Maps (Schaeffer)

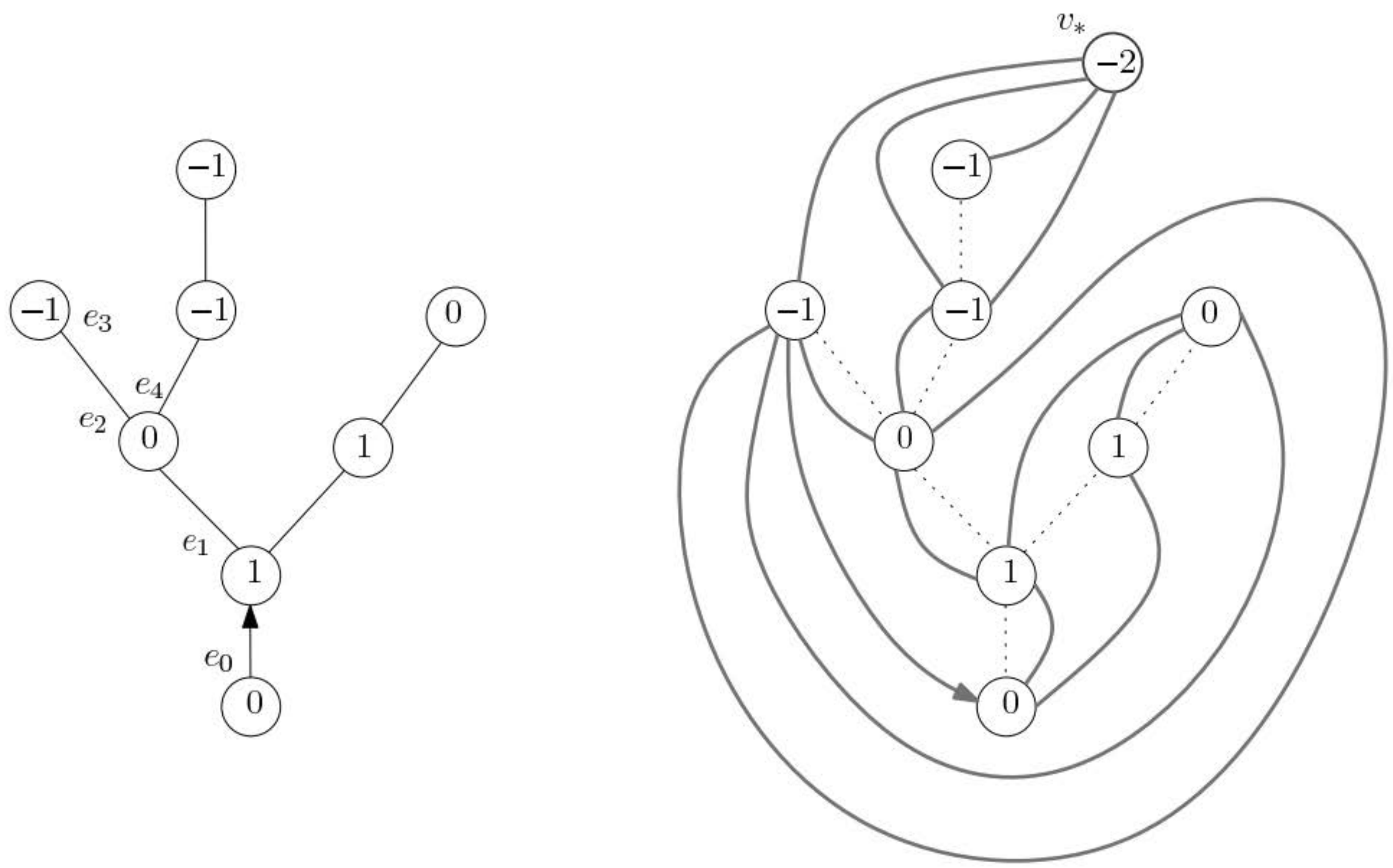
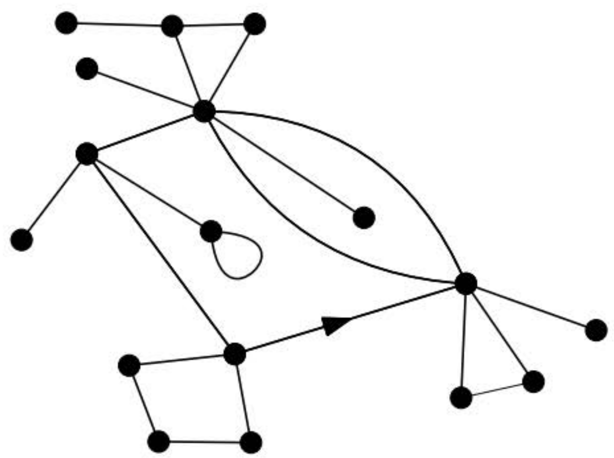
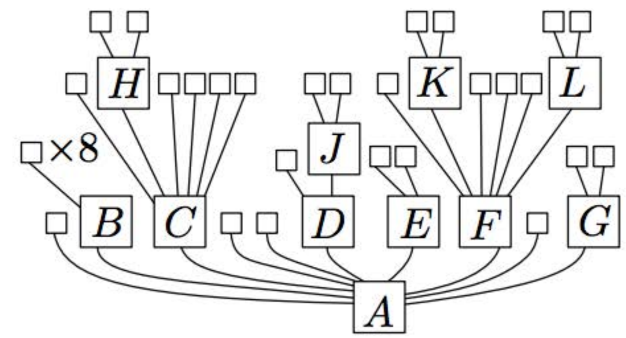


FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

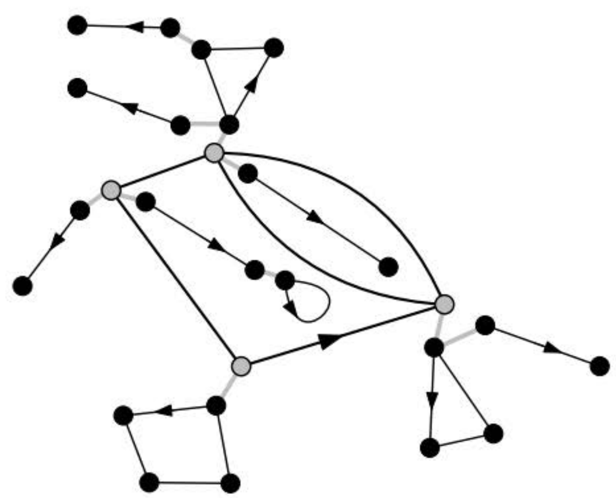
Maps (Addario-Berry)



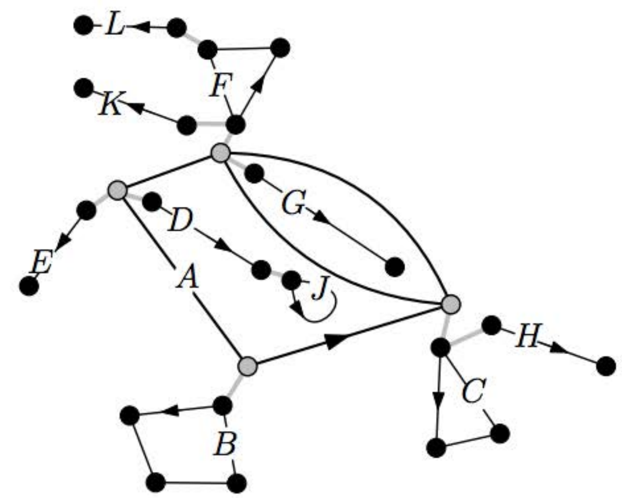
(A) A map M.



(B) The tree T_M. Tiny squares represent trivial blocks.

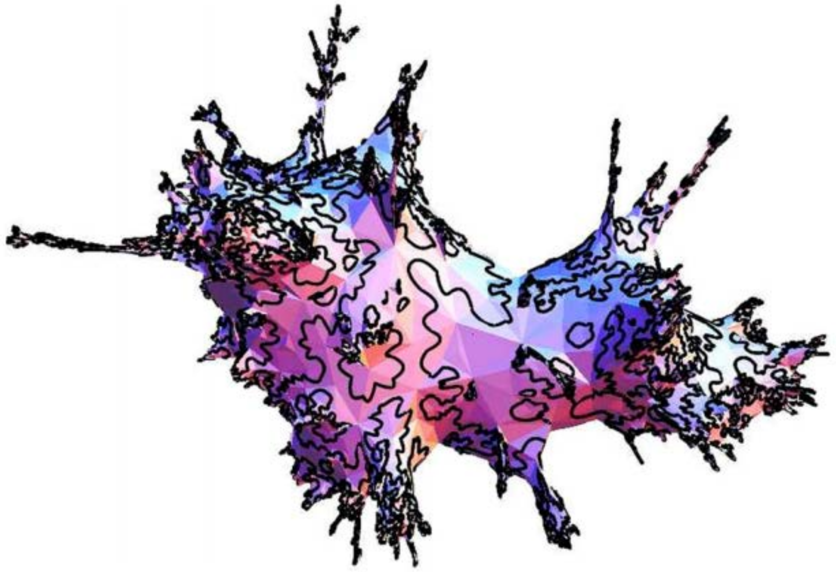


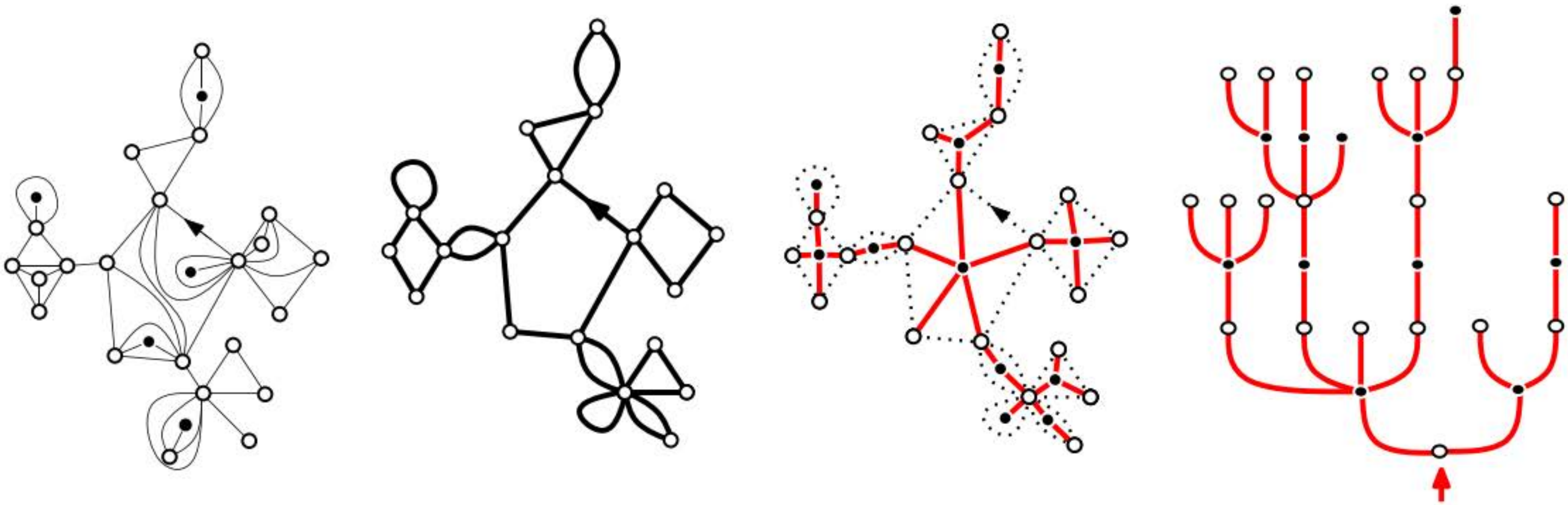
(C) The decomposition of M into blocks. Blocks are joined by grey lines according to the tree structure. Root edges of blocks are shown with arrows.



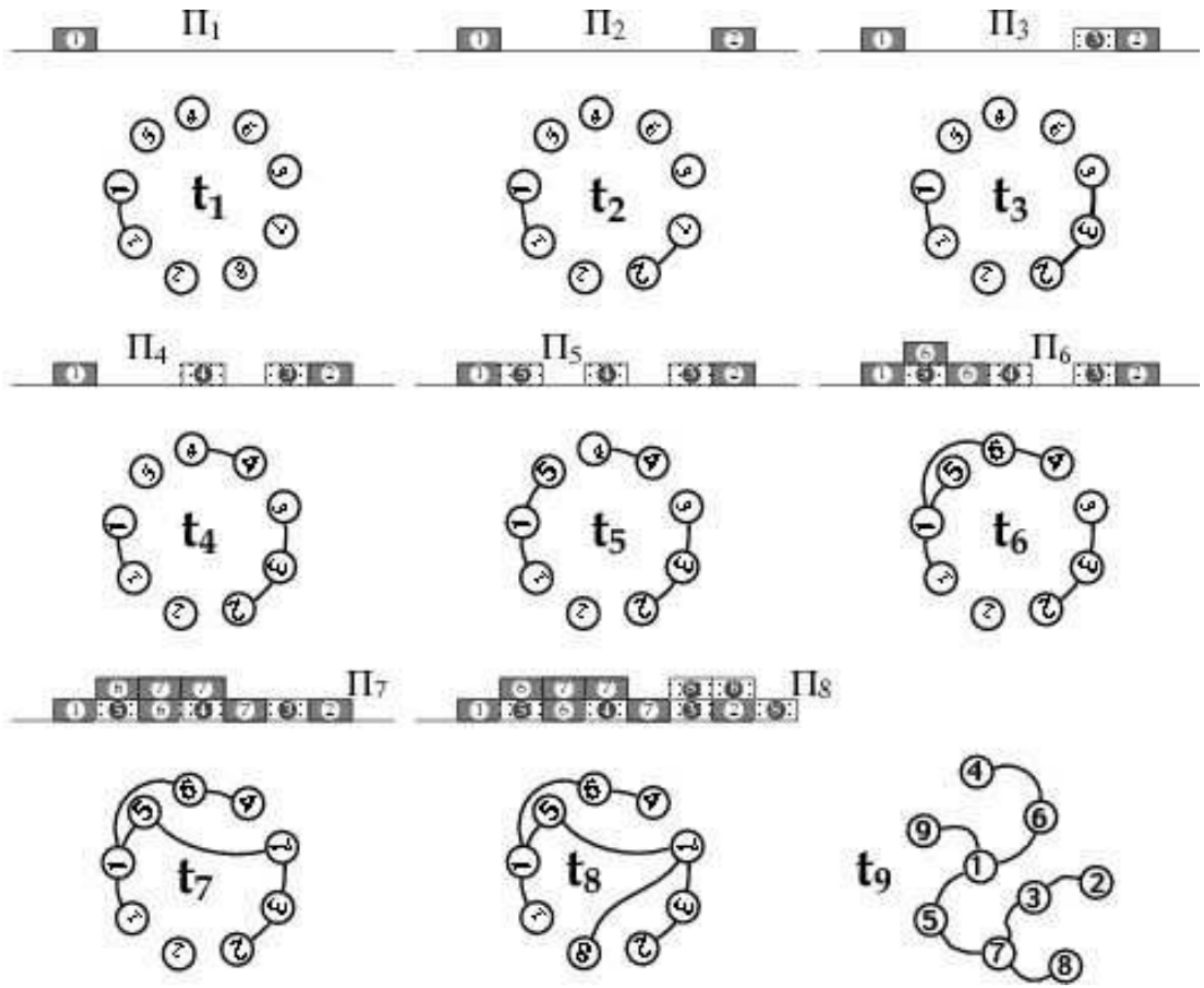
(D) The correspondence between blocks and nodes of T_M. Non-trivial blocks receive the alphabetical label (from A through L) of the corresponding node.

Maps with percolation (Curien, K.)





Parking functions (Chassaing, Louchard)



I. Models coded by trees

II. Local limits of Bienaymé trees

III. Scaling limits of Bienaymé trees

Recall that in a Bienaymé tree, every individual has a random number of children (independently of each other) distributed according to μ (offspring distribution).

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What does a large Bienaymé tree look like, near the root?

Local limits: critical case

Let μ be a **critical** offspring distribution. Let $\mathcal{T}_n$ be a Bienaymé tree conditioned on having n vertices.

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Theorem (Kesten '87, Janson '12, Abraham & Delmas '14)

The convergence

$$\mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_\infty$$

holds in distribution for the local topology, where $\mathcal{T}_\infty$ *is the infinite Bienaymé tree conditioned to survive.*

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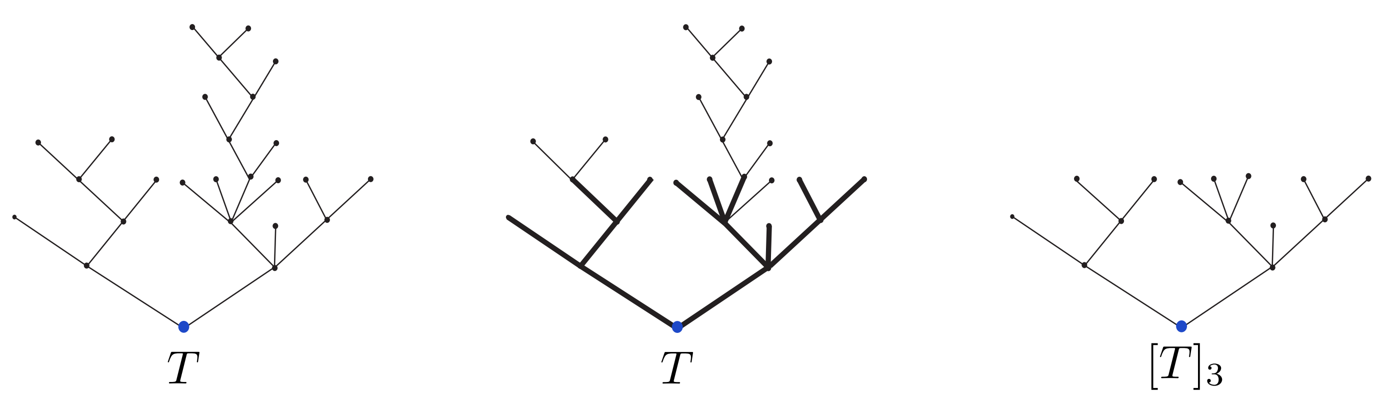
⇝ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of T obtained by keeping the first k children on the first k generations:



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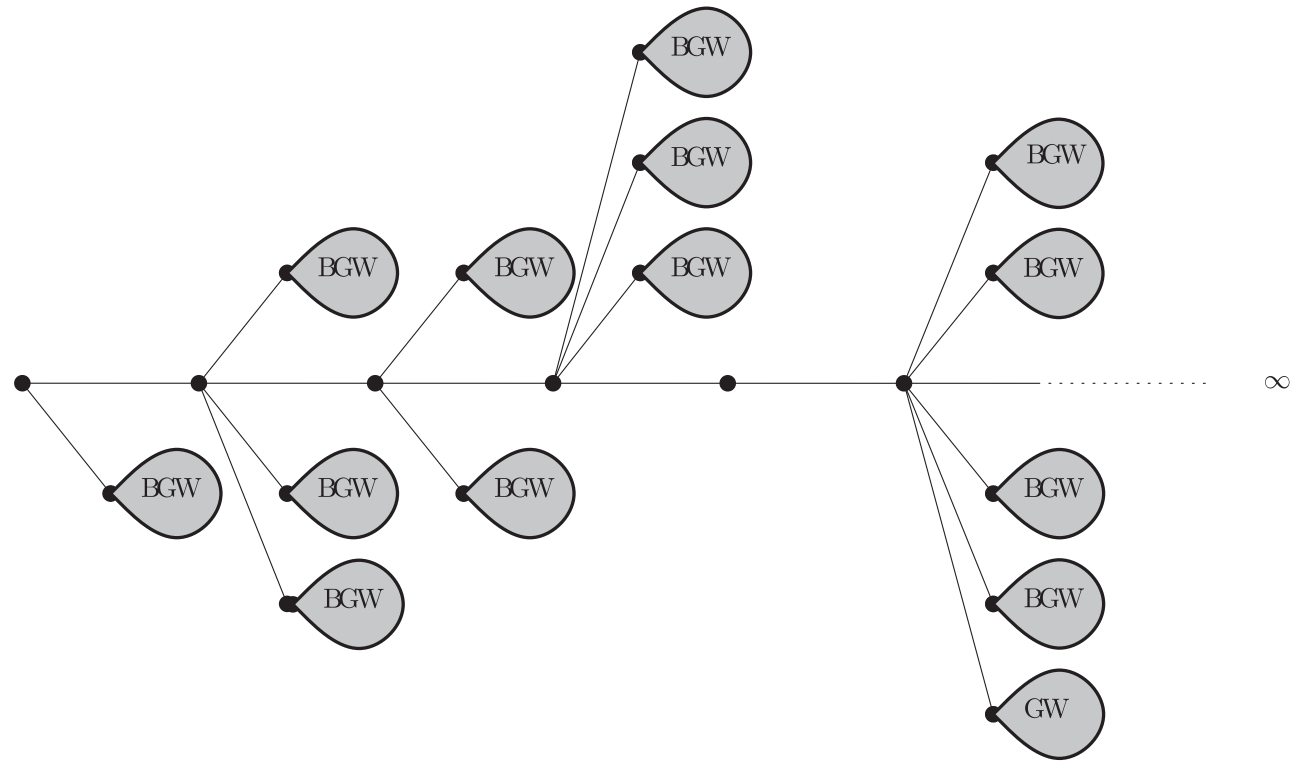

Local limits: subcritical case

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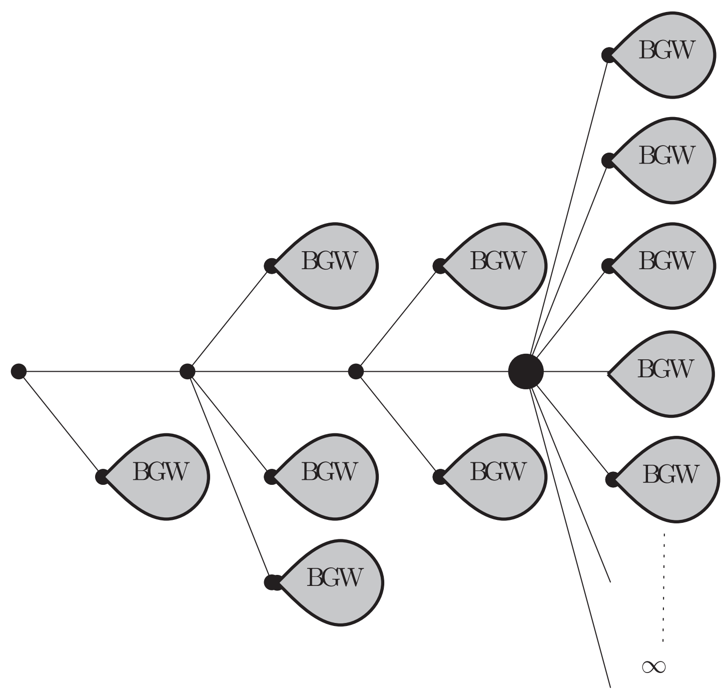

I. Models coded by trees

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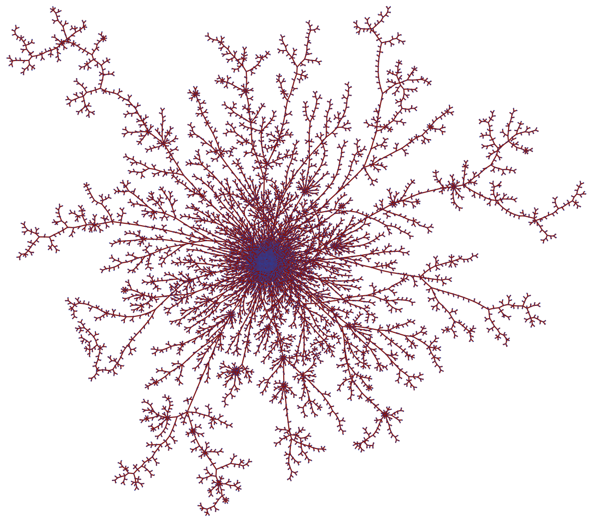
What does a large Bienaymé tree look like, globally?

I have simulated and drawn a uniform plane tree with 10000 vertices. What did I get?



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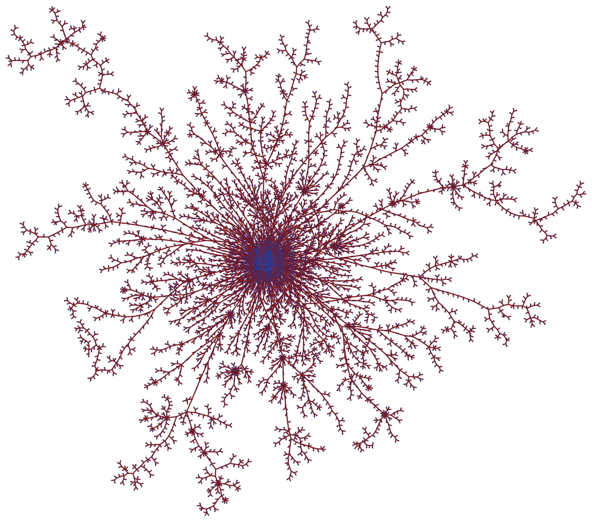

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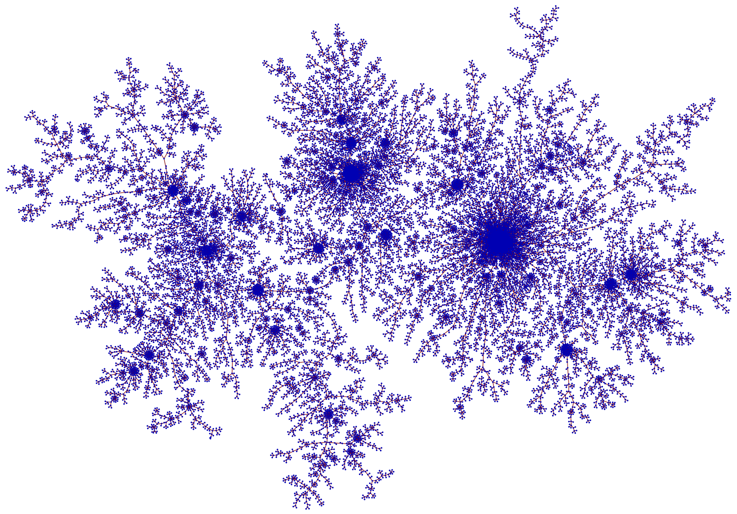

Figure: Result 2.

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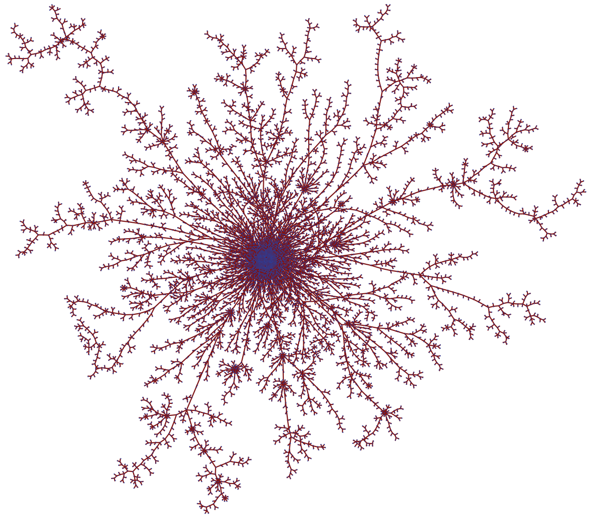

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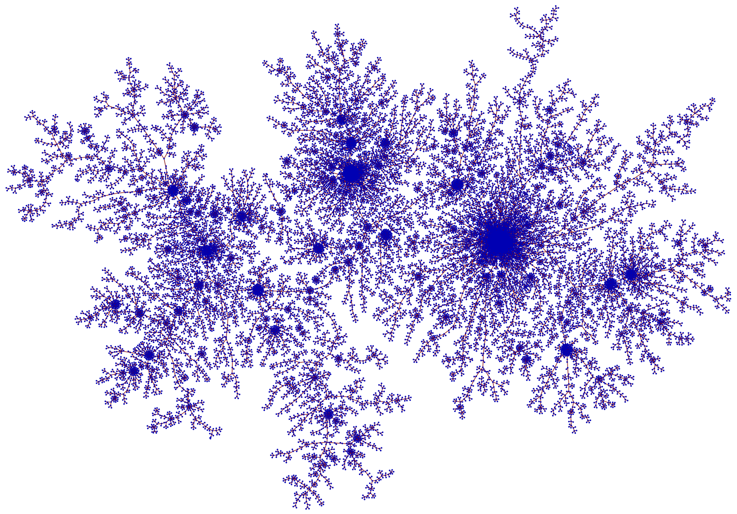

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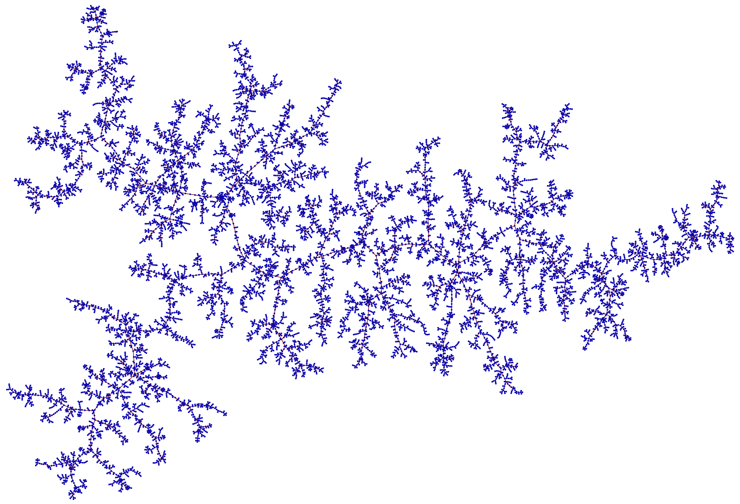

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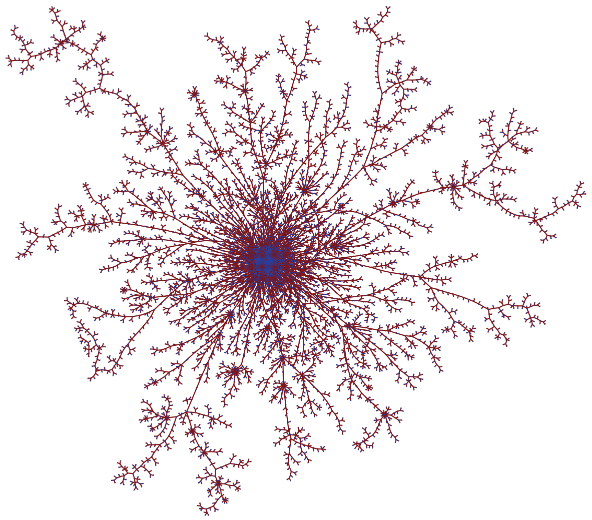

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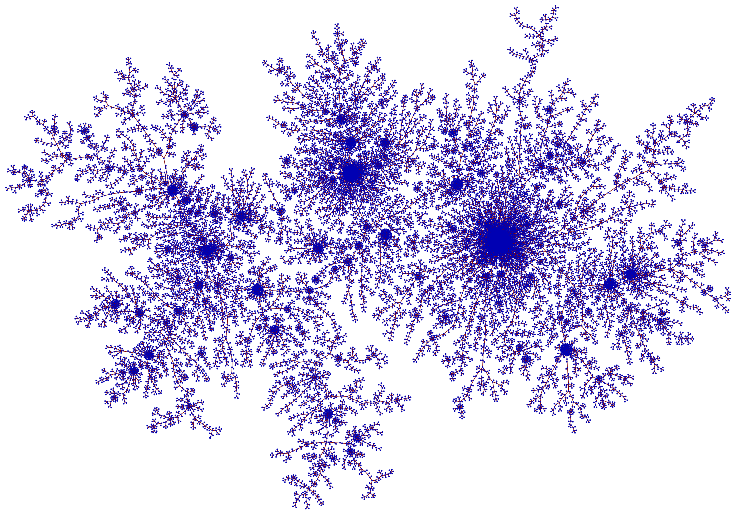

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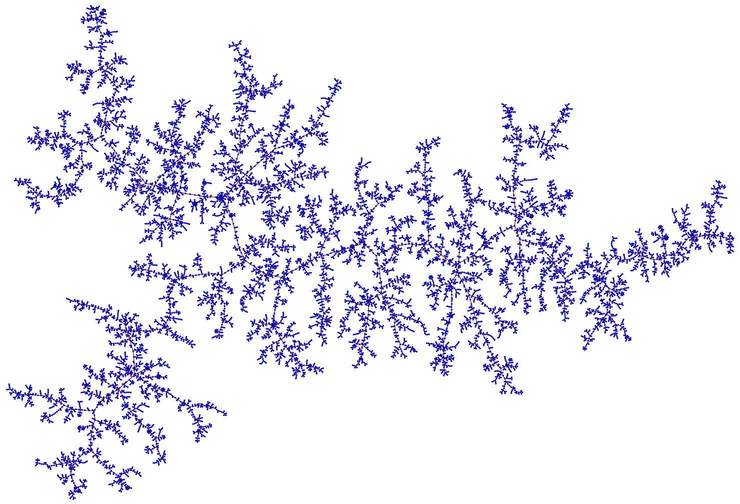

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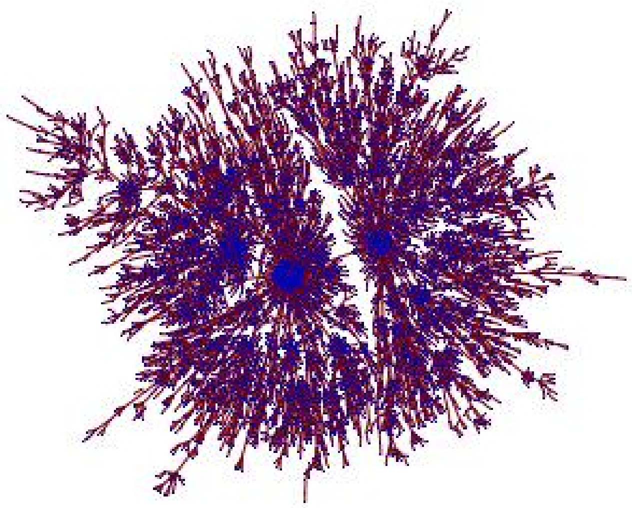

Figure: Result 4.

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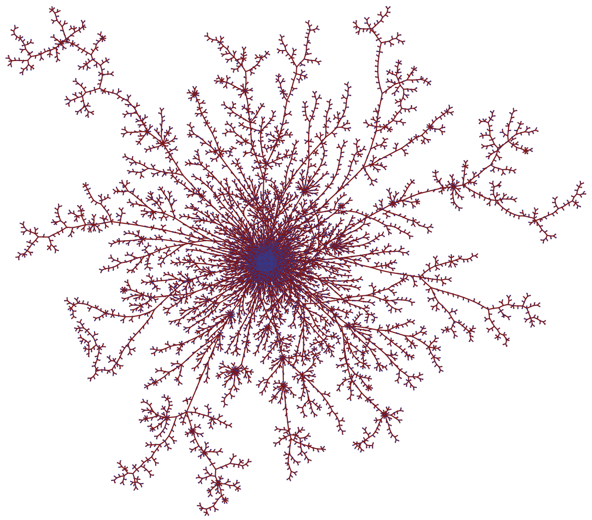

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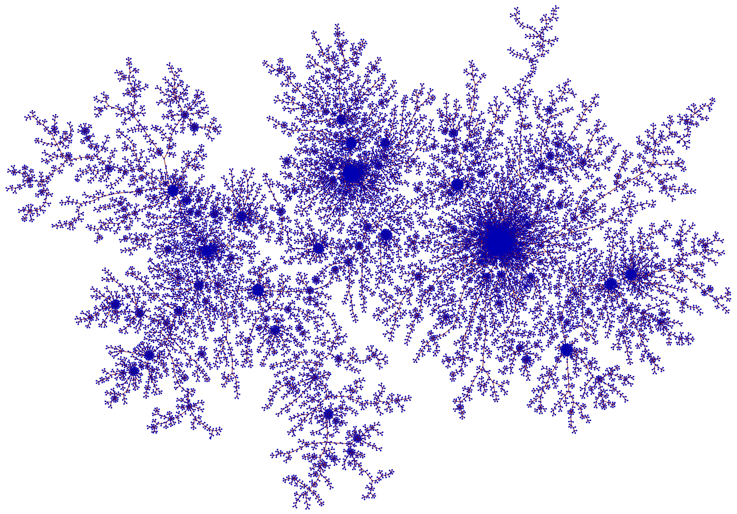

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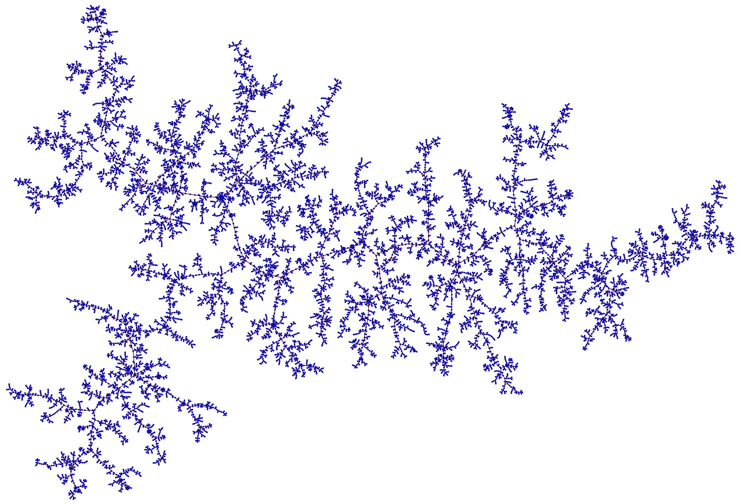

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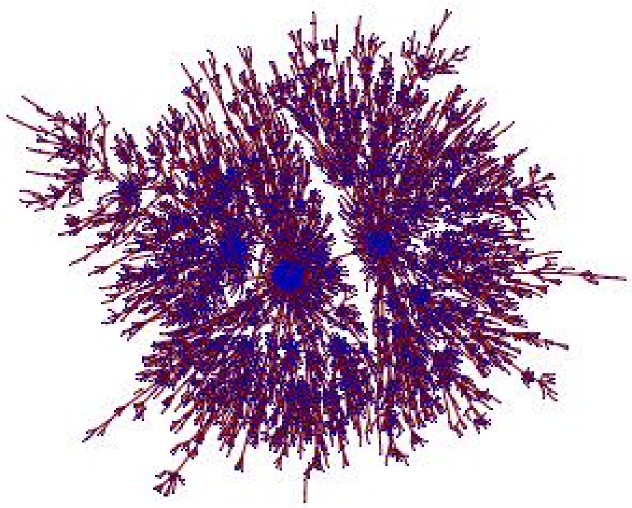

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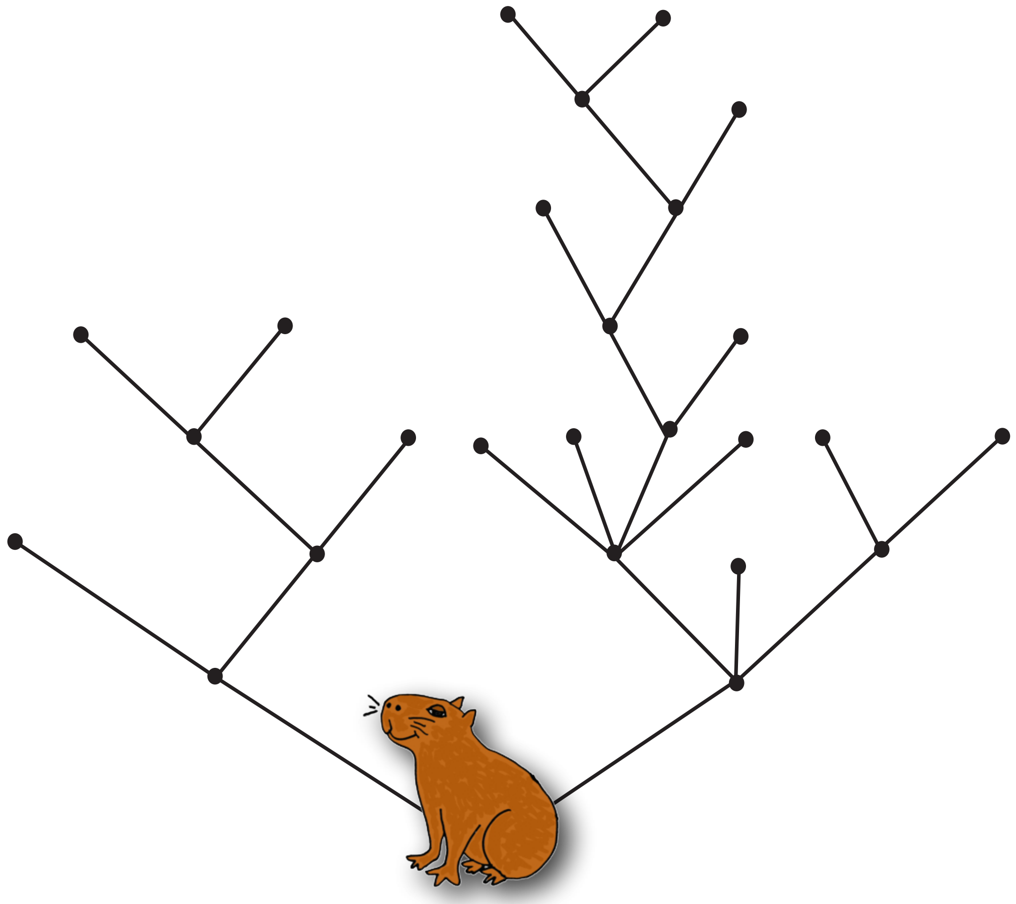
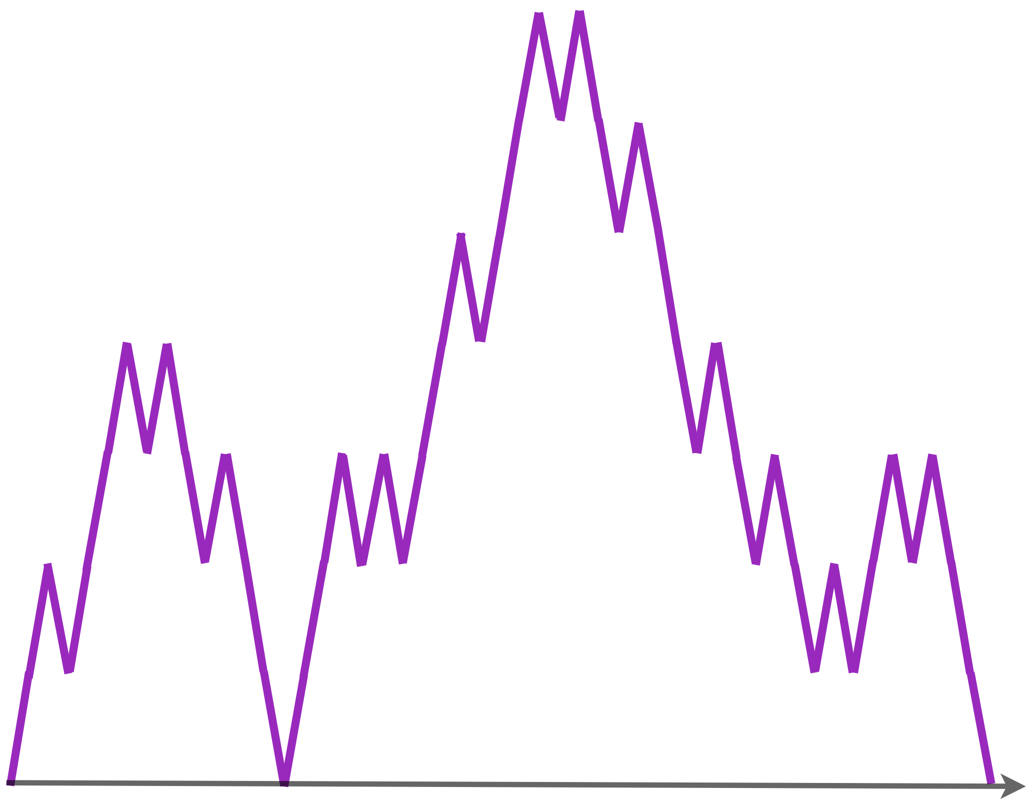
⇝ wooclap.com ; code **nancy2023**.

Coding trees by functions

Contour function of a tree

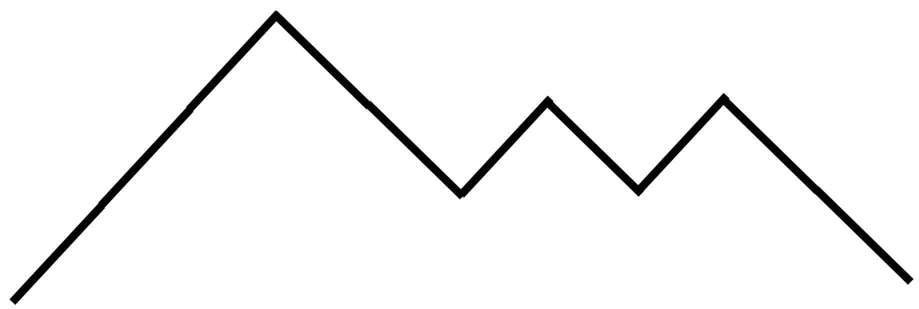
Define the contour function of a tree:





Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



Scaling limits

Scaling limits: finite variance

Let μ be an offspring distribution with **finite** positive variance such that $\sum_{i \geqslant 0} i\mu(i) = 1$. Let $\mathcal{T}_n$ be a Bienaymé tree conditioned on having n vertices.

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Theorem (Aldous '93)

Let σ^2 *be the variance of* μ*. Let* $t \mapsto C_t(\mathcal{T}_n)$ *be the contour function of* $\mathcal{T}_n$*. Then:*

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n)\right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n \to \infty]{(d)}$$

where the convergence holds in distribution in $\mathcal{C}([0,1], \mathbb{R})$

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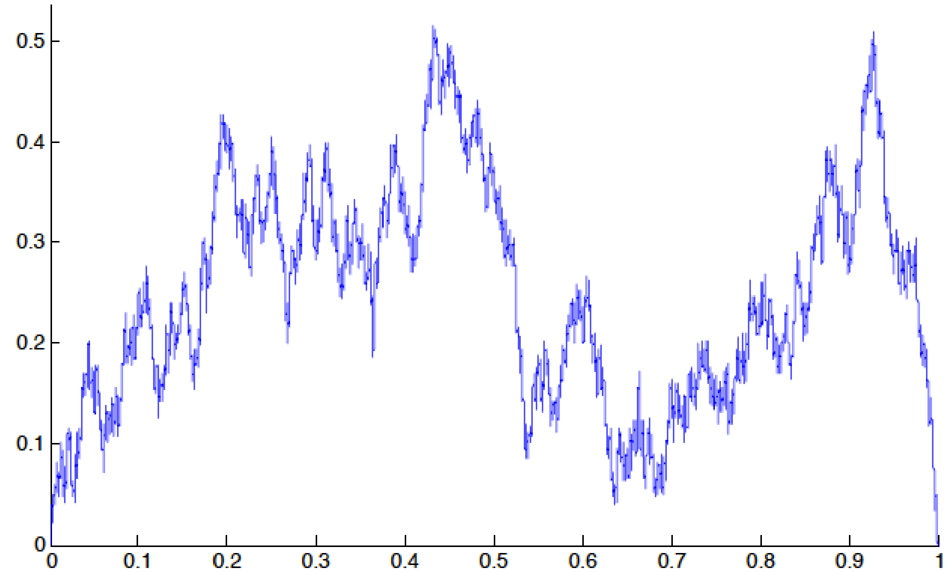

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Consequence: for every $a > 0$,

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Let μ be an offspring distribution with **finite** positive variance such that $\sum_{i \geqslant 0} i\mu(i) = 1$. Let $\mathcal{T}_n$ be a Bienaymé tree conditioned on having n vertices.

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Let σ^2 be the variance of μ. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:

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Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

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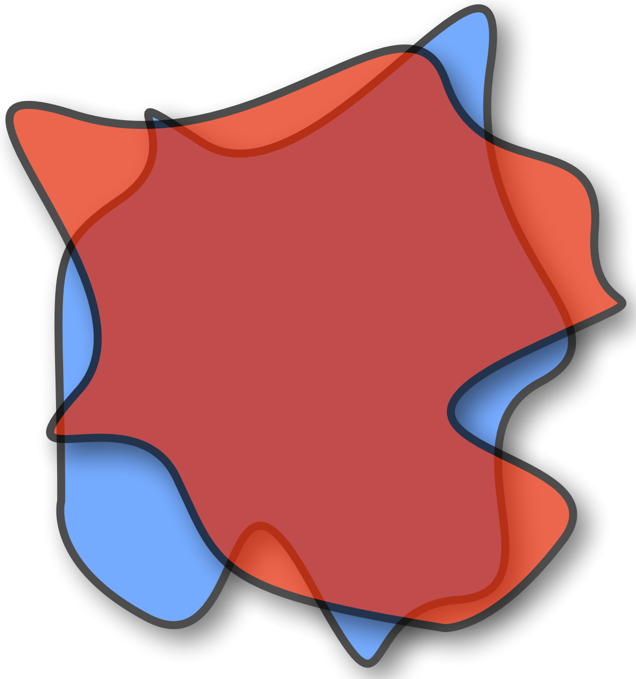
$$d_H(X, Y) = \inf\{r > 0; X \subset Y_r \text{ and } Y \subset X_r\}.$$



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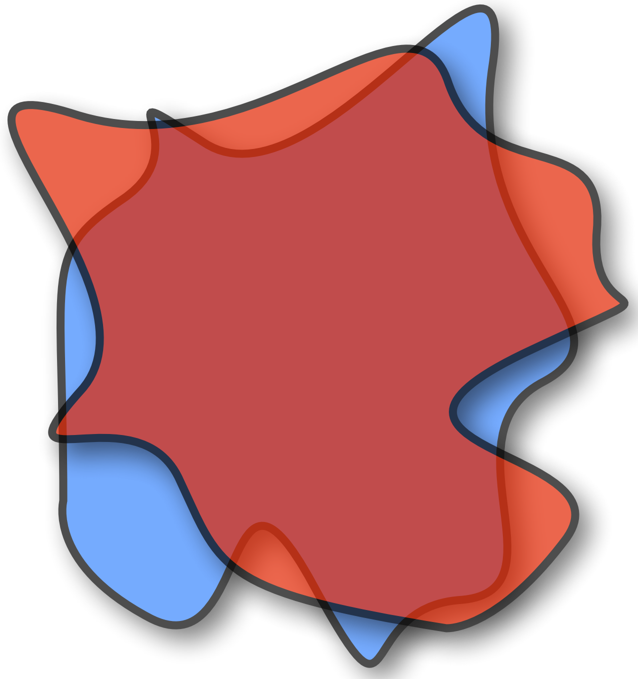

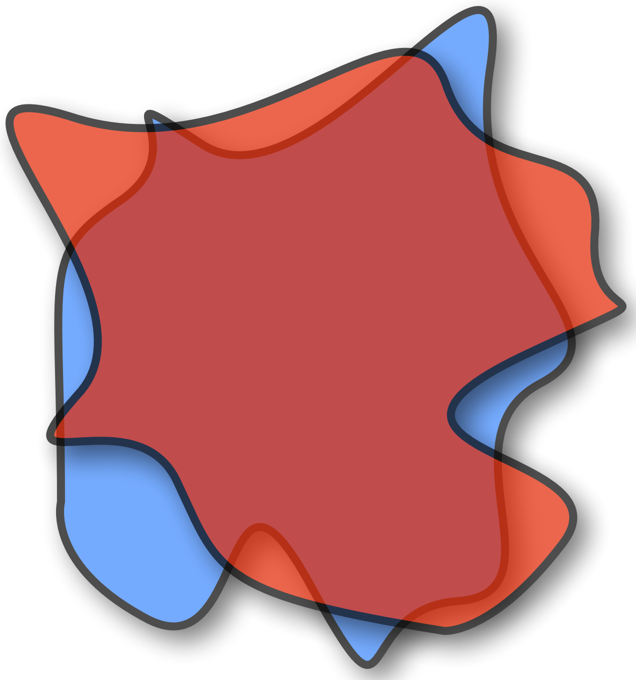

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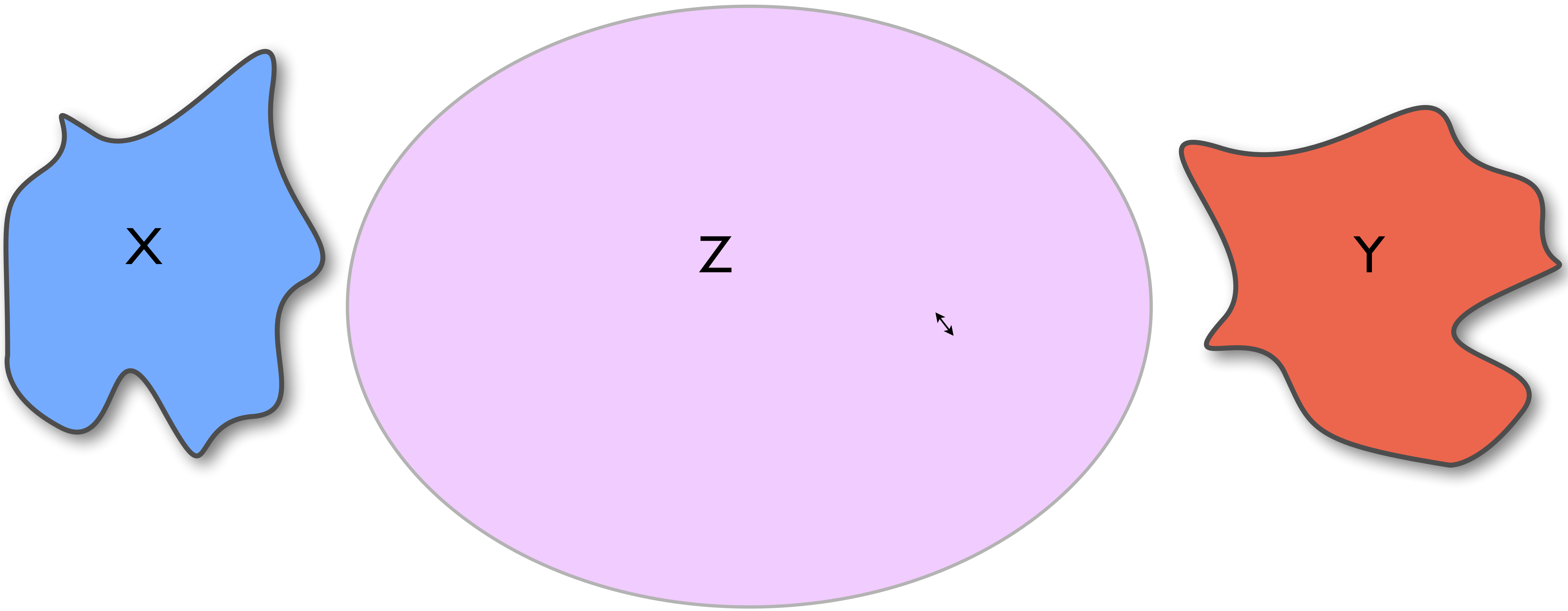

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

⇝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

$$\frac{\sigma}{2\sqrt{n}} \cdot \mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_{\mathbb{e}},$$

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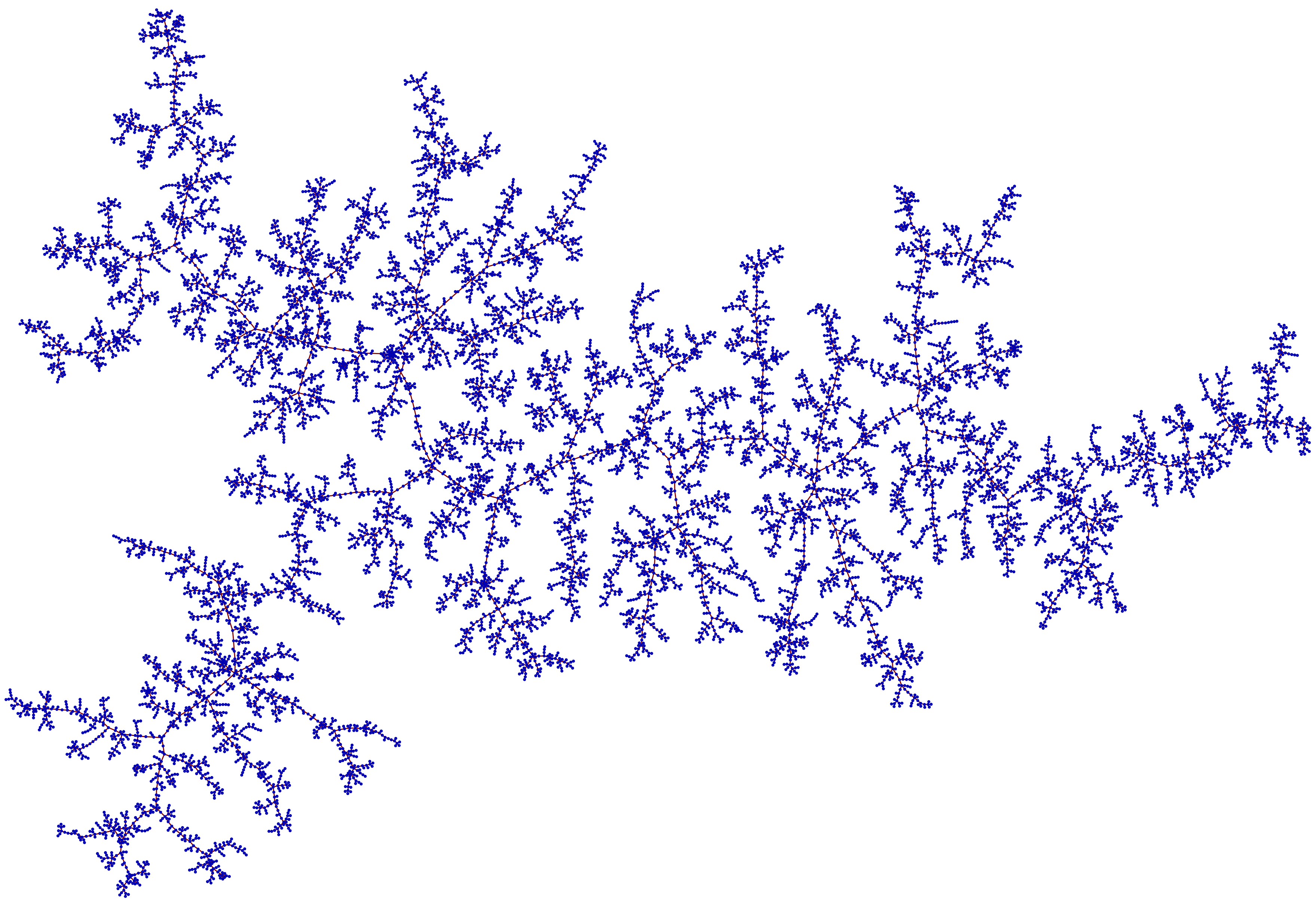
The metric space $\mathcal{T}_{\mathbb{e}}$ is called the *Brownian continuum random tree (CRT)*, and is coded by a Brownian excursion.



An approximation of a realization of a Brownian CRT

What about non-critical offspring distributions?

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- wooclap.com ; code **nancy2023**.

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4. Because we condition on total population size, the distribution of $\mathscr{T}_n$ is unchanged by replacing ξ with another distribution χ in the same exponential family

$$P(\xi = i) = c\theta^i P(\chi = i), \qquad i \geq 0 \text{ for some } c, \theta.$$

Thus there is no essential loss of generality in considering only critical branching processes.

Condensation (subcritical case)

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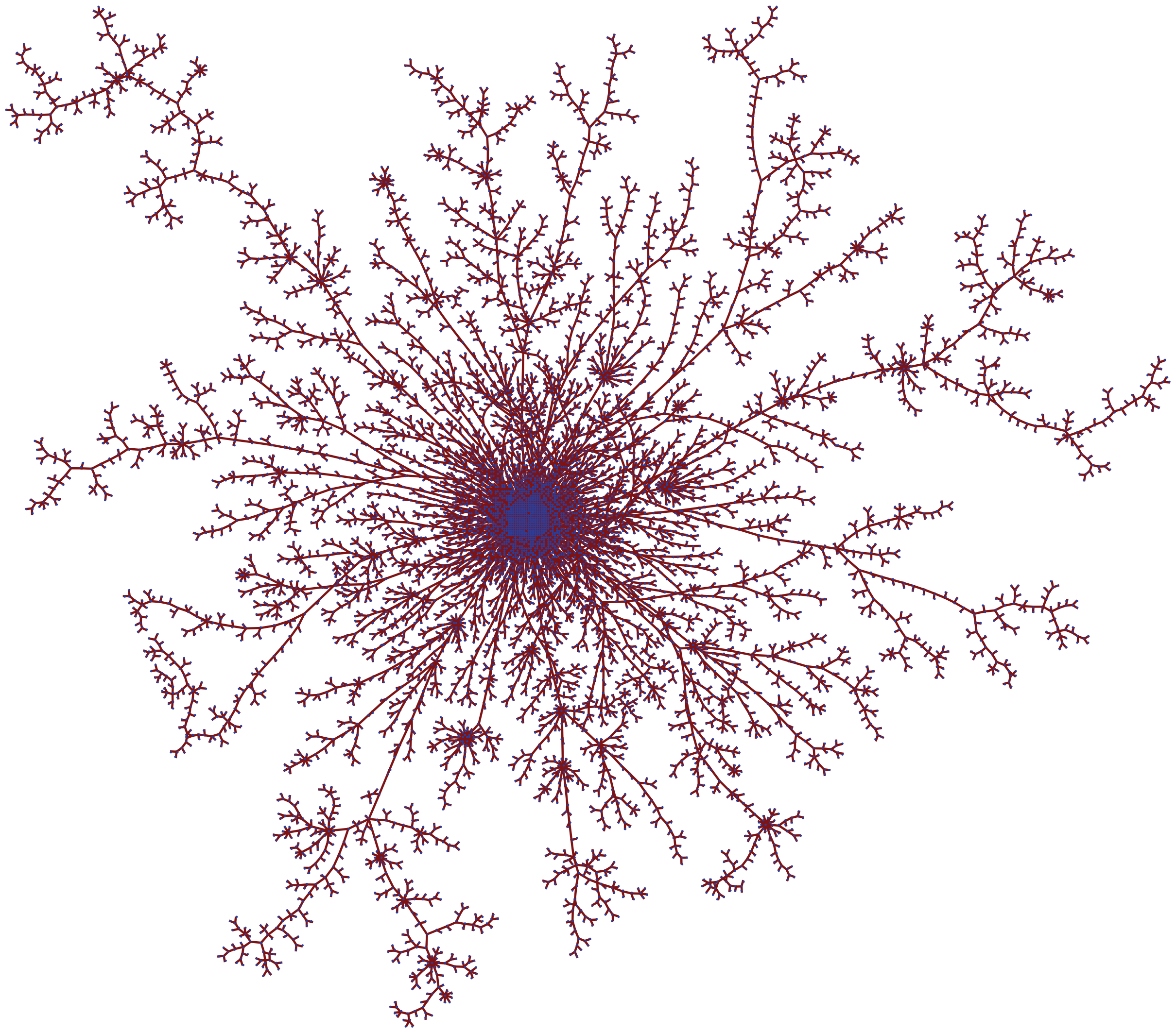

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Let m *be the mean of* μ*. Denote by* $\Delta(\mathcal{T}_n)$ *the maximum degree of* $\mathcal{T}_n$*. Then*

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- Are there scaling limits?