The Brownian triangulation: a universal limit for random plane non-crossing configurations

(joint work with Nicolas Curien)

Igor Kortchemski (Université Paris-Sud, Orsay, France)

MIT Probability Seminar, April 2nd 2012

Motivations

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What is the sense of the convergence when the objects are **random**?
$\rightarrow$ **Convergence in distribution**

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We write:

$$X_n \quad \xrightarrow[n \to \infty]{(d)} \quad X$$

Outline

I. The discrete object

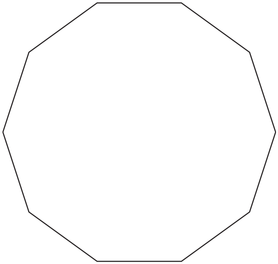
II. The limiting continuous object

III. Proving the convergence

IV. Application to the study of uniform dissections

I. The discrete objects

Let P_n be the polygon whose vertices are $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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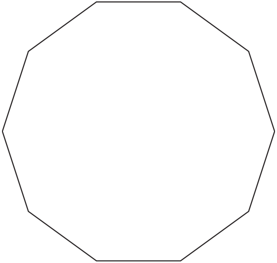

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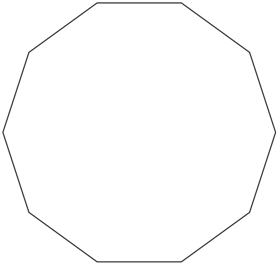

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What happens for n large?

Case of dissections of P_n.

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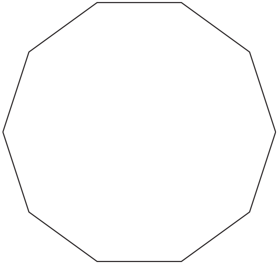

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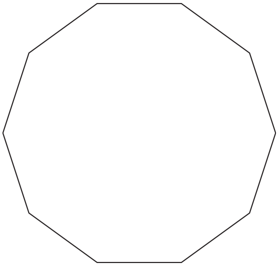

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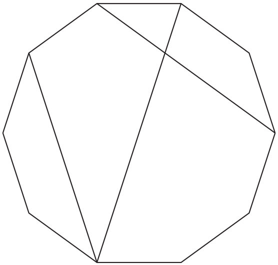

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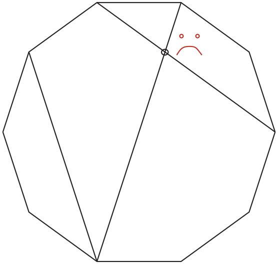

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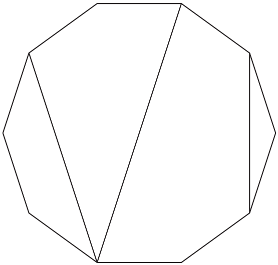

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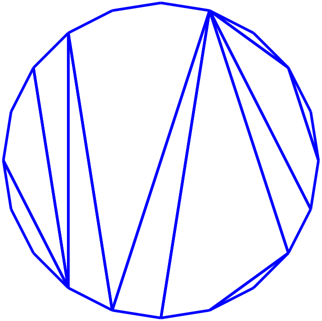
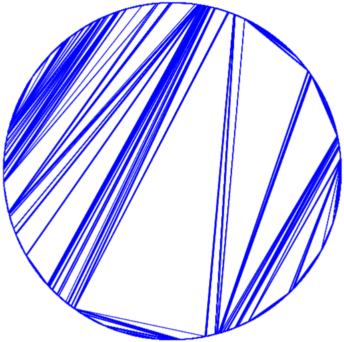


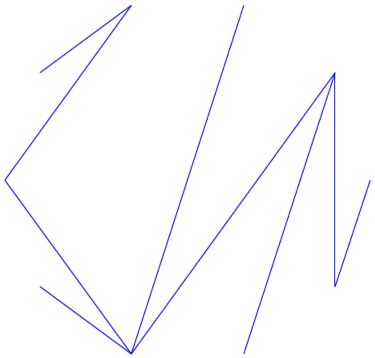


Samples of $\mathcal{D}_{18}$ and $\mathcal{D}_{15000}$.

Case of non-crossing trees of P_n.

Non-crossing trees

Example of a non-crossing tree of P_{10}:



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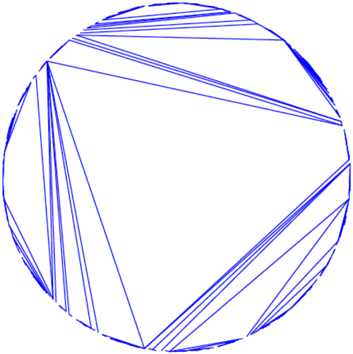
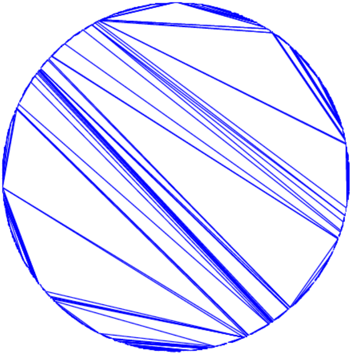


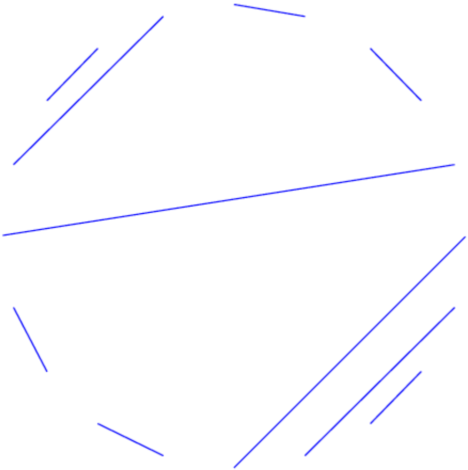


Samples of $\mathcal{T}_{500}$ and $\mathcal{T}_{1000}$.

Case of non-crossing pair-partitions of P_{2n} .

Non-crossing pair partitions

Example of a non-crossing pair-partition of P_{20}:



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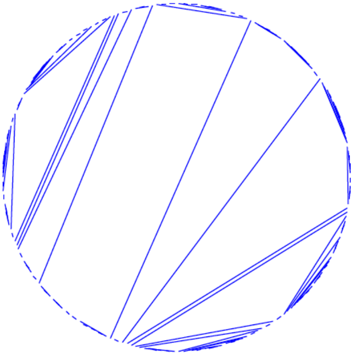

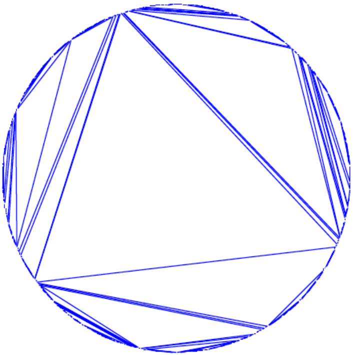

Samples of $\mathcal{Q}_{250}$ and $\mathcal{Q}_{1000}$.

History of non-crossing configurations of P_n

Combinatorical point of view:

- Counting and bijections for non-crossing trees: Dulucq & Penaud (1993), Noy (1998), ...
- Counting of various non-crossing configurations: Flajolet & Noy (1999)

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- Uniform triangulations (maximal degree): Devroye, Flajolet, Hurtado, Noy & Steiger (1999) et Gao & Wormald (2000)
- Non-crossing trees (total length, maximal degree): Deutsch & Noy (2002), Marckert & Panholzer (2002)
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Geometrical point of view:

- Aldous (1994): large uniform triangulations
- K' (2011): dissections with large faces (non uniform)

II. Construction of the continuous limiting object:

the Brownian triangulation (Aldous, '94)

Interlude: Brownian motion and the Brownian excursion

Brownian motion

Theorem (Donsker)

Let $(X_n)_{n \geqslant 1}$ *be a sequence of i.i.d random variables with* $\mathbb{E}[X_1] = 0$ *and* $\sigma^2 = \mathbb{E}\left[{X_1}^2\right] < \infty$.

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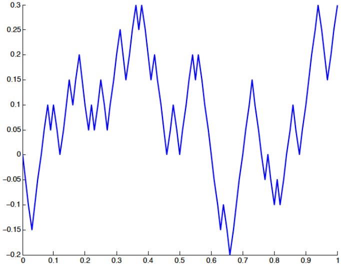

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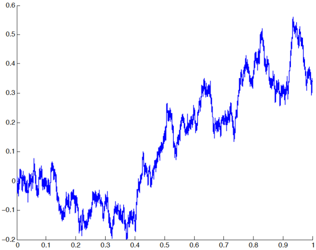

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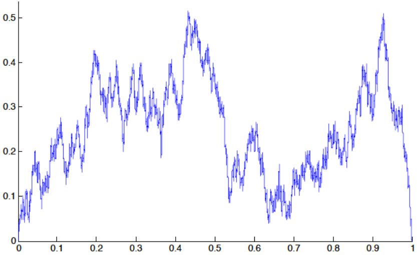

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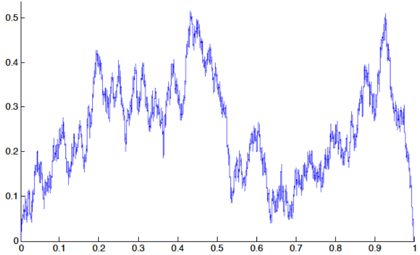

The Brownian excursion can be seen as Brownian motion $(W_t, 0 \leqslant t \leqslant 1)$ conditioned on $W_1 = 0$ and $W_t > 0$ for $t \in (0, 1)$.

Construction of the limiting object

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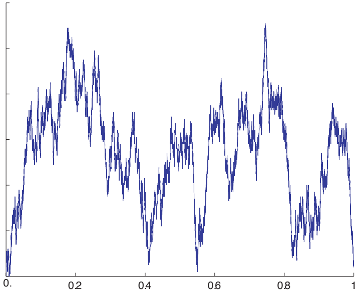

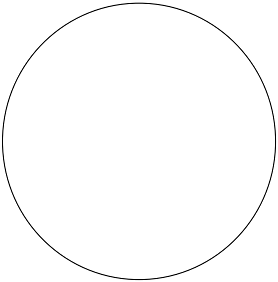

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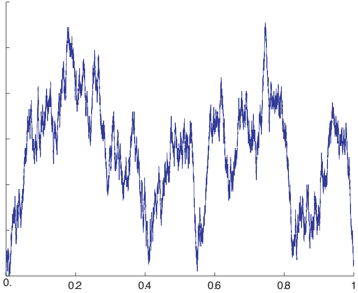

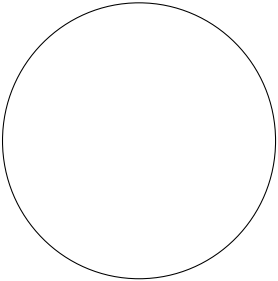

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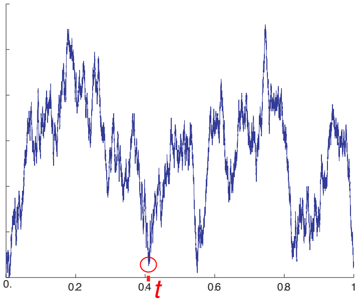

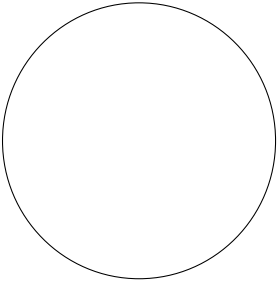

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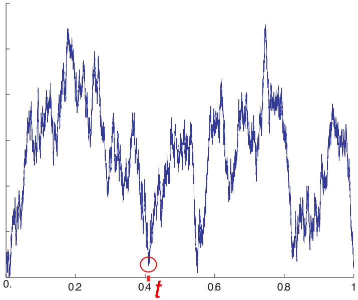

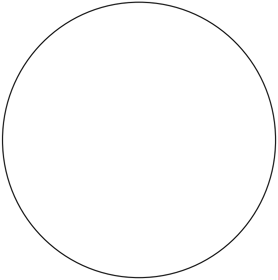

Let t be a local minimum time. Set $g_t = \sup\{s < t;\ \mathbb{e}_s = \mathbb{e}_t\}$ and $d_t = \inf\{s > t;\ \mathbb{e}_s = \mathbb{e}_t\}$.

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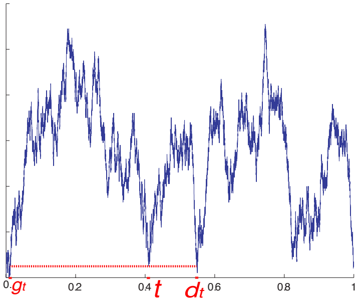

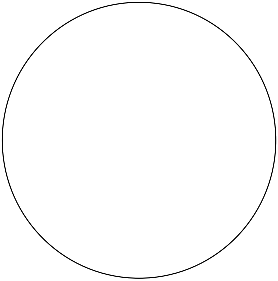

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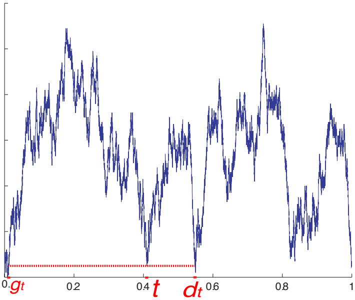

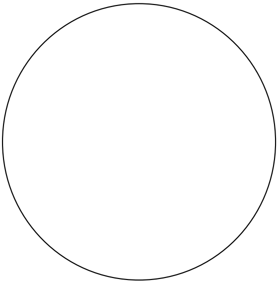

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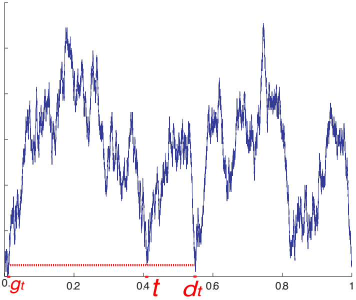

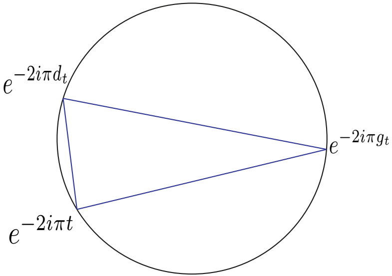

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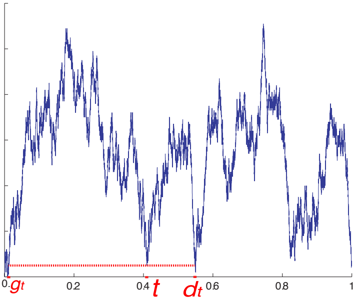

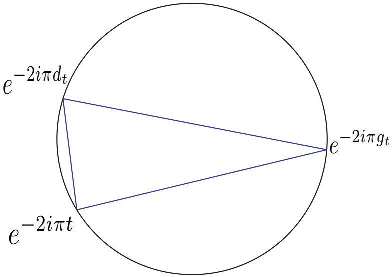

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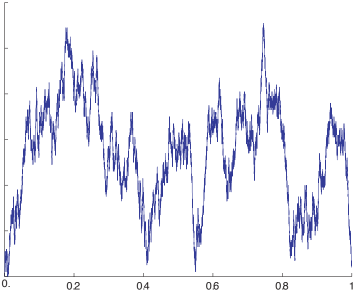

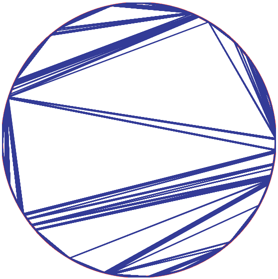

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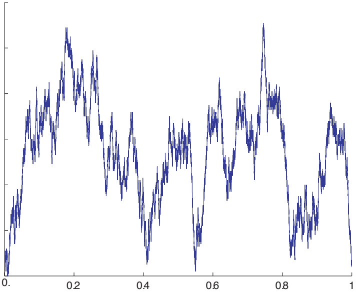

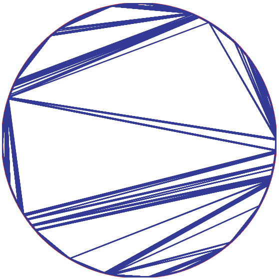

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The closure of the set thus obtained, denoted by $L(\mathbb{e})$, is called the Brownian triangulation.

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- There exists a "stable" analog of $L(\mathbb{e})$ with big holes (K. '11).

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- The area of the largest face of χ_n converges in distribution towards the area of the largest triangle of $L(\mathbb{e})$.

III. How does one establish the convergence of all these non-crossing uniformly distributed models towards the Brownian triangulation?

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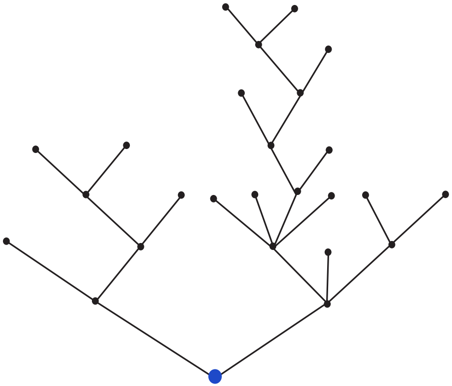
Key point: Each one of the previous models can be coded by a conditioned Galton-Watson tree.

Coding trees



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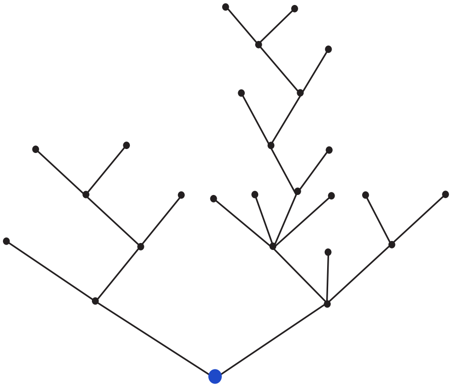

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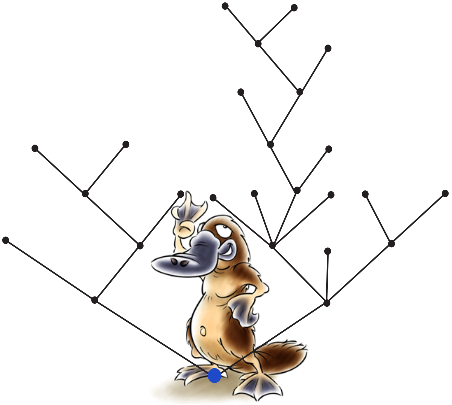

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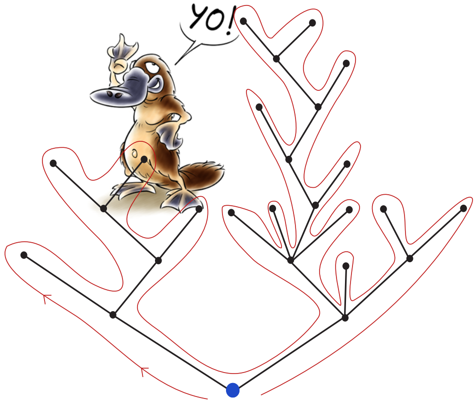

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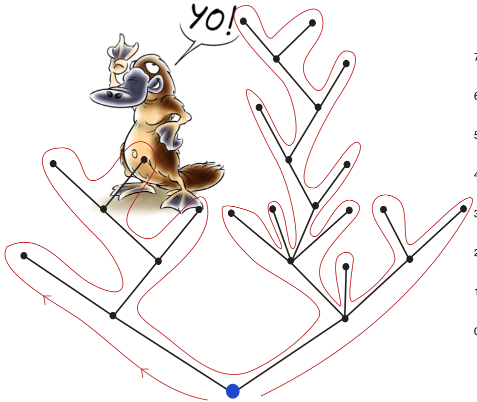

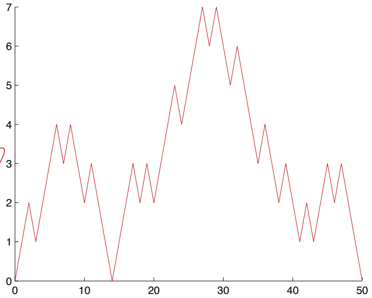

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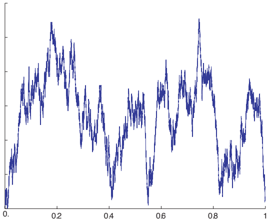
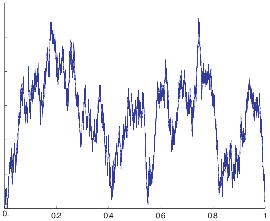


Scaled contour function of a large conditioned Galton-Watson tree.



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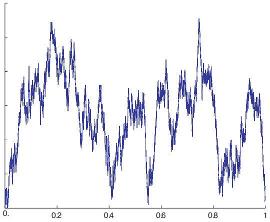

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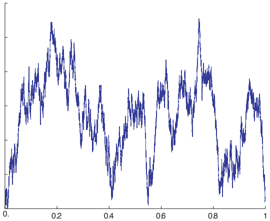

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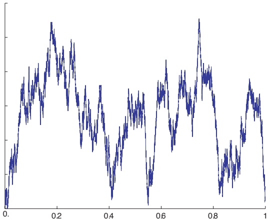

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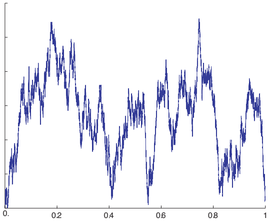

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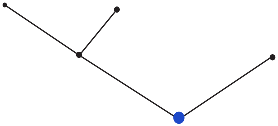
It follows that the non-crossing uniformly distributed models converge towards $L(\mathbb{e})$.

Brief recap on Galton-Watson trees

We consider rooted plane (oriented) trees.



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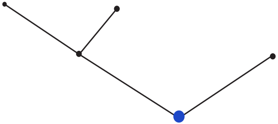

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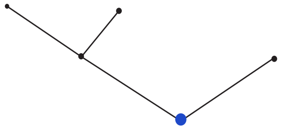

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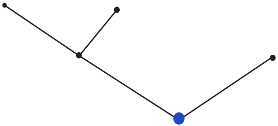

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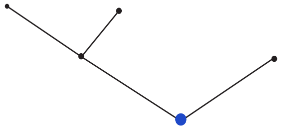

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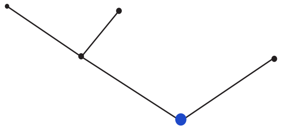

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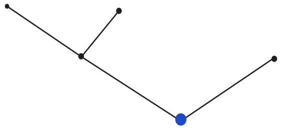

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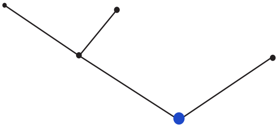

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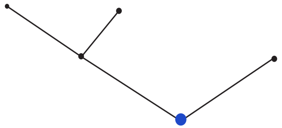

Here, $\zeta(\tau) = 5$ and $\lambda(\tau) = 3$.

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Proposition

Let ν be defined by $\nu(k) = 1/2^{k+1}$ for $k \geqslant 0$. Then the law of a uniformly distributed tree with n vertices is the law of a GW_ν tree conditioned on having n vertices.

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Proof.

Let τ be a tree with n vertices. It suffices to prove that $\mathbb{P}_\nu[\tau]$ depends only on n.

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Let ν be defined by $\nu(k) = 1/2^{k+1}$ for $k \geqslant 0$. Then the law of a uniformly distributed tree with n vertices is the law of a GW_ν tree conditioned on having n vertices.

Proof.

Let τ be a tree with n vertices. It suffices to prove that $\mathbb{P}_\nu[\tau]$ depends only on n. We have (k_u being the number of children of u):

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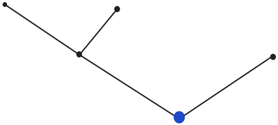

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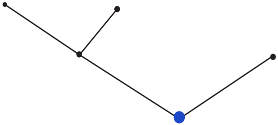

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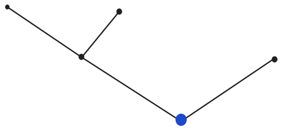
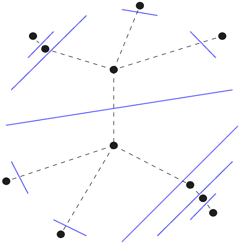


How can one code non-crossing uniformly distributed models by a conditioned Galton-Watson tree?

Coding uniform pair-partitions by Galton-Watson trees.

Consider the dual of a uniform non-crossing pair-partition of P_{2n}:



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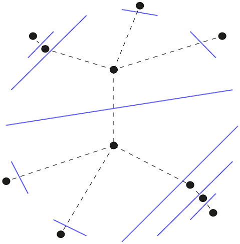

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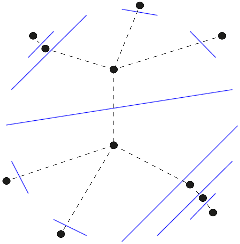

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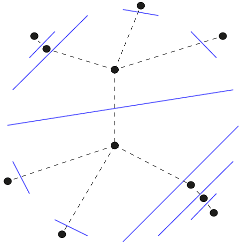

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Theorem (Aldous '93)

Let $\mathfrak{t}_n$ be a random tree distributed according to $\mathbb{P}_{Geom(1/2)}[\,\cdot\,|\,\zeta(\tau)=n+1]$. Let σ^2 be the variance of $Geom(1/2)$. Then:

$$\left(\frac{\sigma}{2\sqrt{n}}C_{2nt}(\mathfrak{t}_n), 0 \leqslant t \leqslant 1\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbb{e}_t, 0 \leqslant t \leqslant 1).$$

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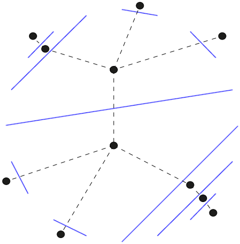

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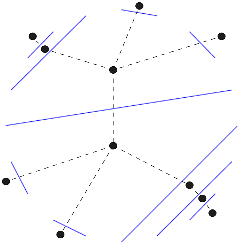

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It follows that uniform non-crossing pair-partitions of P_{2n} converge towards the Brownian triangulation.

Coding uniform dissections by Galton-Watson trees.

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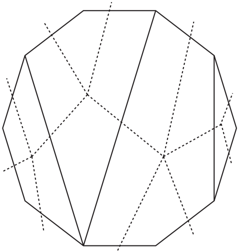

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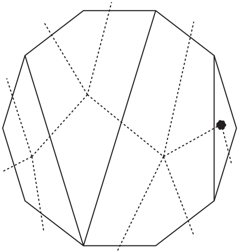

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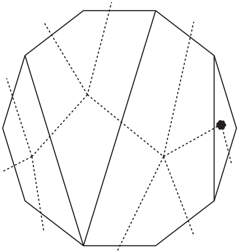

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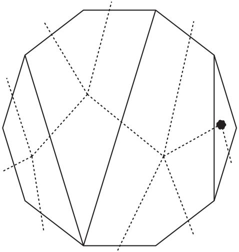

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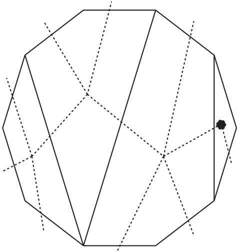

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It follows that uniform dissections of P_n converge towards the Brownian triangulation.

Conclusion: In these uniform models, some independence is hiding.

IV. Application to the study of uniform dissections

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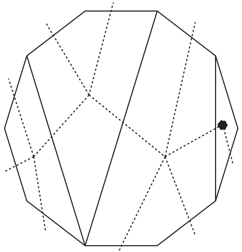

$\mathcal{D}_n$: uniform dissection of P_n. Recall that: the dual of $\mathcal{D}_n$ is a tree with law $\mathbb{P}_{\mu_0}[\cdot \mid \lambda(\tau) = n-1]$, where $(i \geqslant 2)$:

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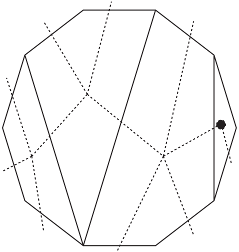

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Theorem (Flajolet & Noy '99)

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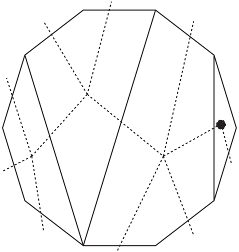

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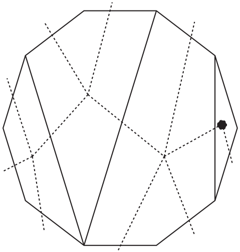

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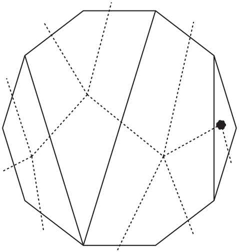

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Set $\beta = 2+\sqrt{2}$. *For every* $c > 0$, *we have:*

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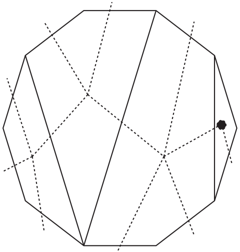

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Let $\partial^{(n)}$ be the number of diagonals ending at the vertex with affix 1 in $\mathcal{D}_n$.

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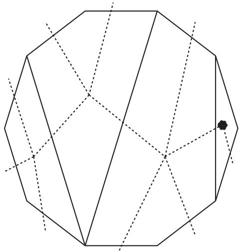

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$$\mathbb{P}(\partial^{(n)} = k) \quad \xrightarrow[n\to\infty]{} \quad (k+1)\mu_0^2(1-\mu_0)^k.$$

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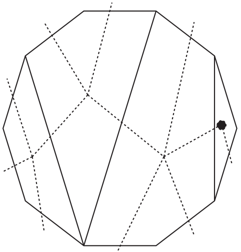

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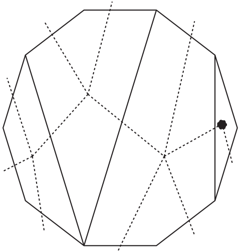

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This is satisfied for another value of b in the case of uniform triangulations (Devroye, Flajolet, Hurtado, Noy & Steiger '99 et Gao & Wormald '00)

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