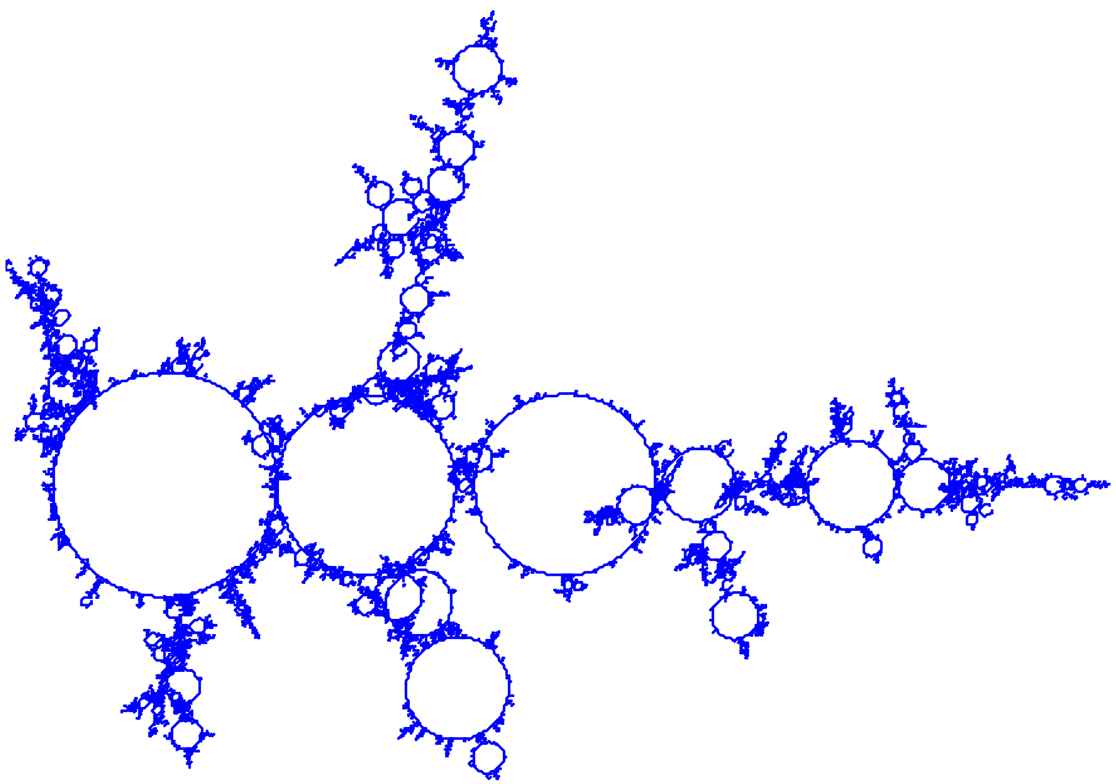
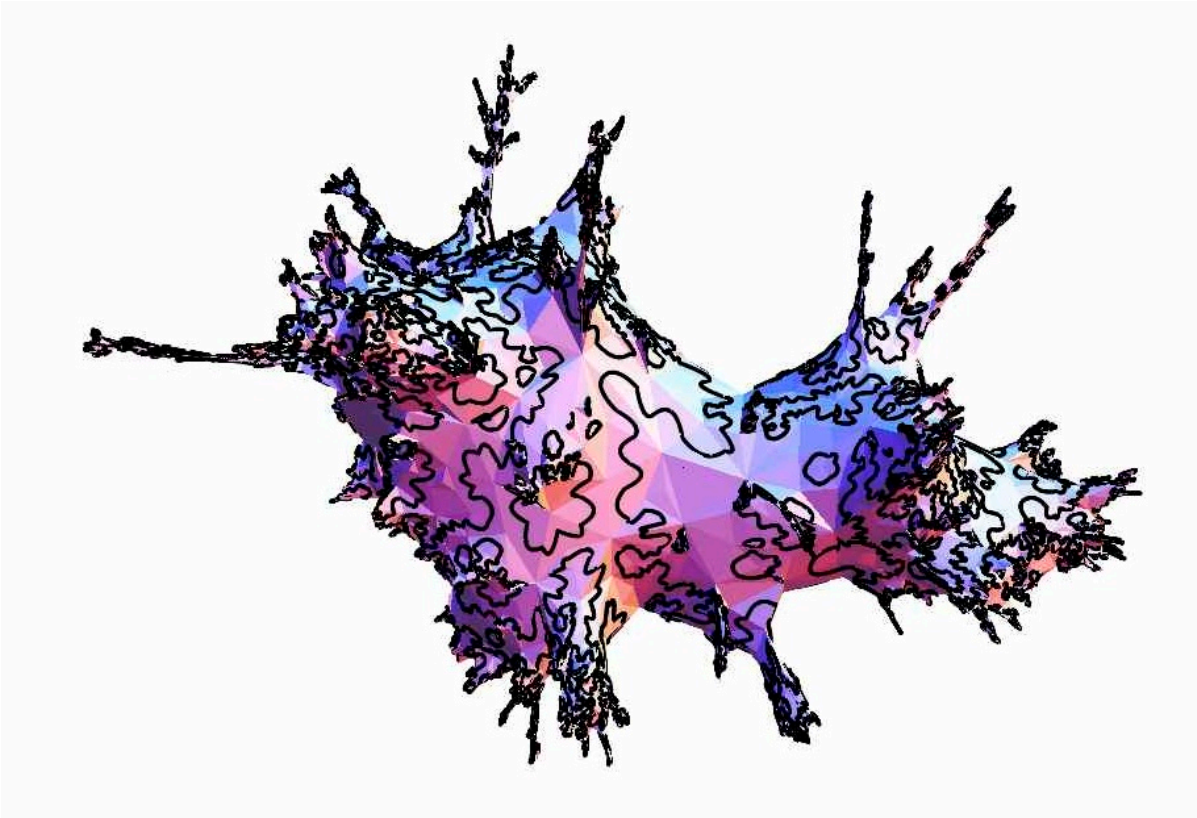
Random stable looptrees and percolation on random maps





Igor Kortchemski (joint work with Nicolas Curien)
Université Zürich

Seminar on Stochastic Processes – Zürich – October 2014

Outline

Motivations

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$\leadsto$ **Convergence in distribution**

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$\leadsto$ **Finite graphs, seen as compact metric spaces**

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$\rightsquigarrow$ **Convergence of compact metric spaces for the Gromov–Hausdorff topology**

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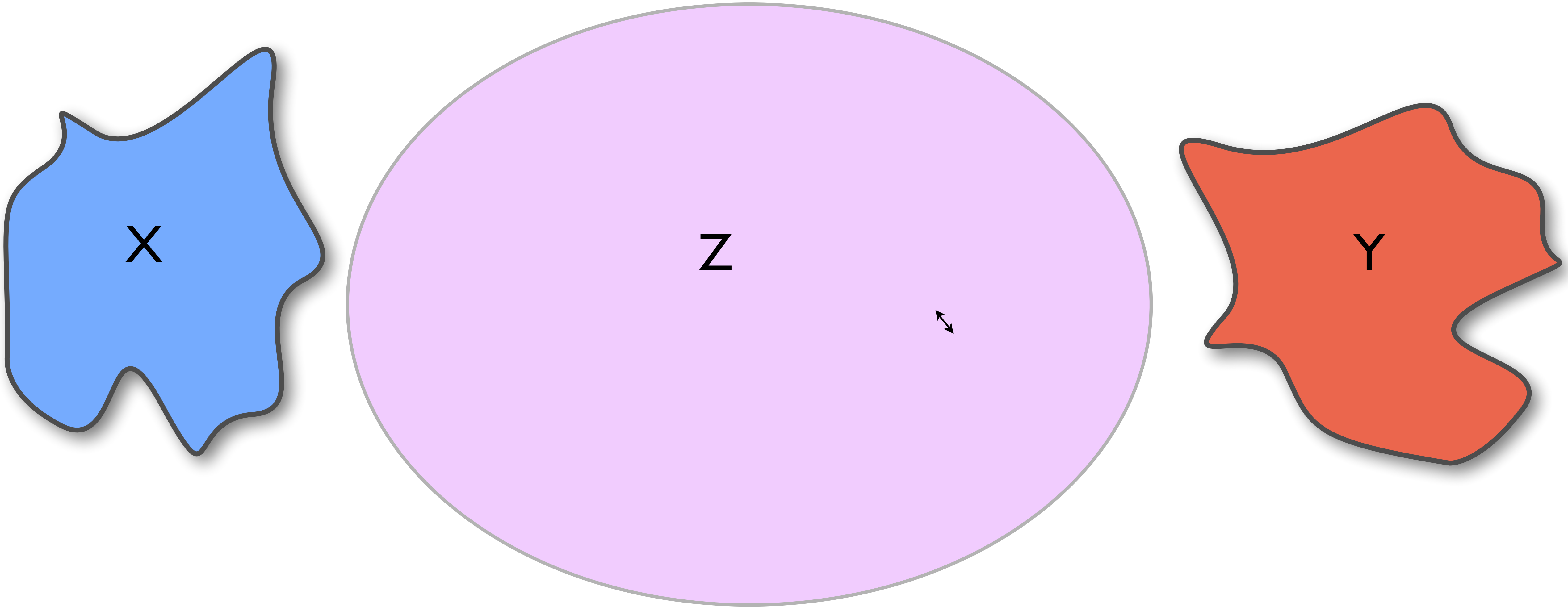

The Gromov–Hausdorff distance between X and Y is the minimal Hausdorff distance between all possible embeddings of X and Y into a common metric space Z.

Motivations

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I. Galton–Watson trees and their scaling limits

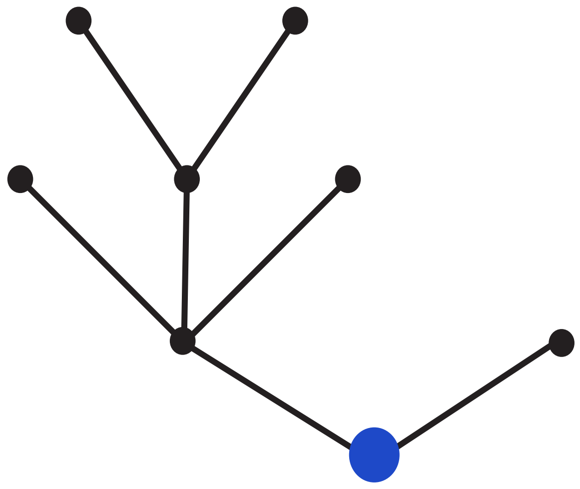
II. Looptrees

III. Looptrees and preferential attachment

IV. Looptrees and percolation on random triangulations

Reminder on Galton–Watson trees

Let ρ be a probability measure on $\{0, 1, 2, \ldots\}$ such that $\sum_i i\rho(i) \leqslant 1$ and $\rho(1) < 1$.



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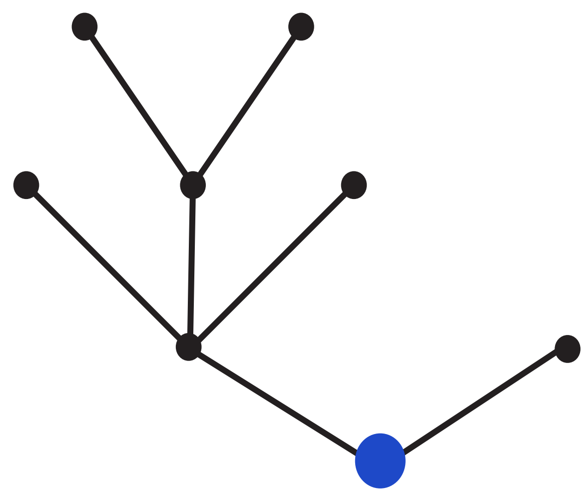

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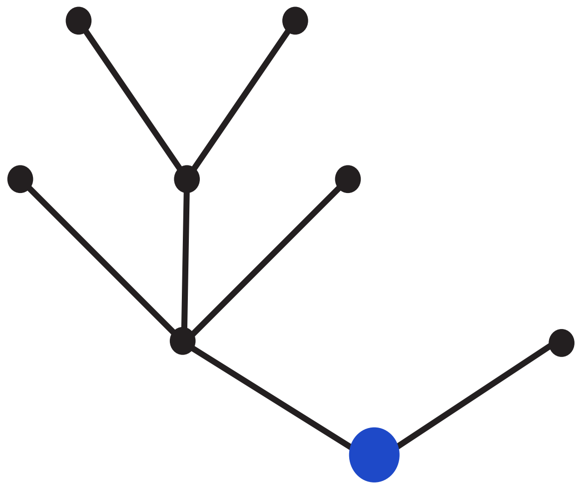

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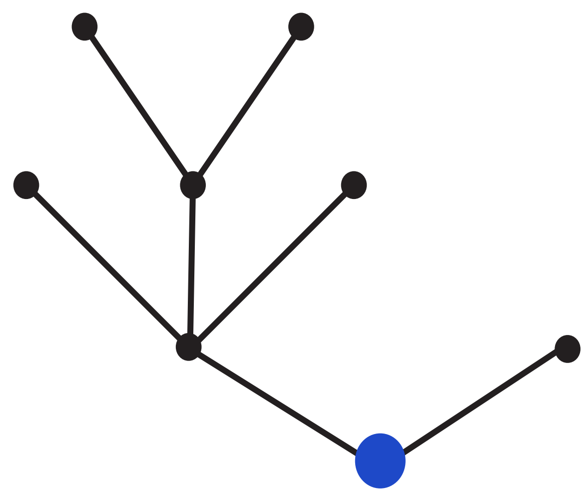

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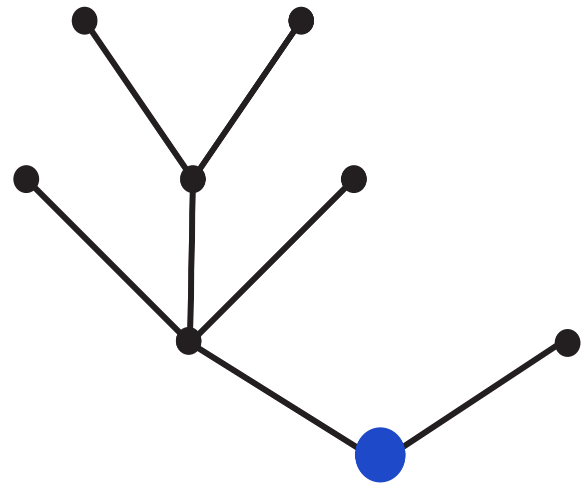

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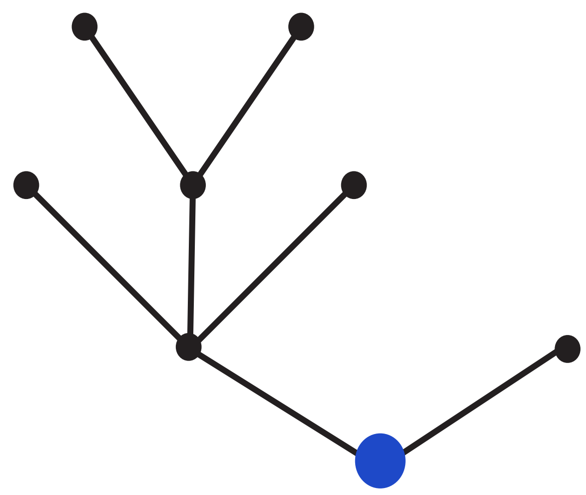

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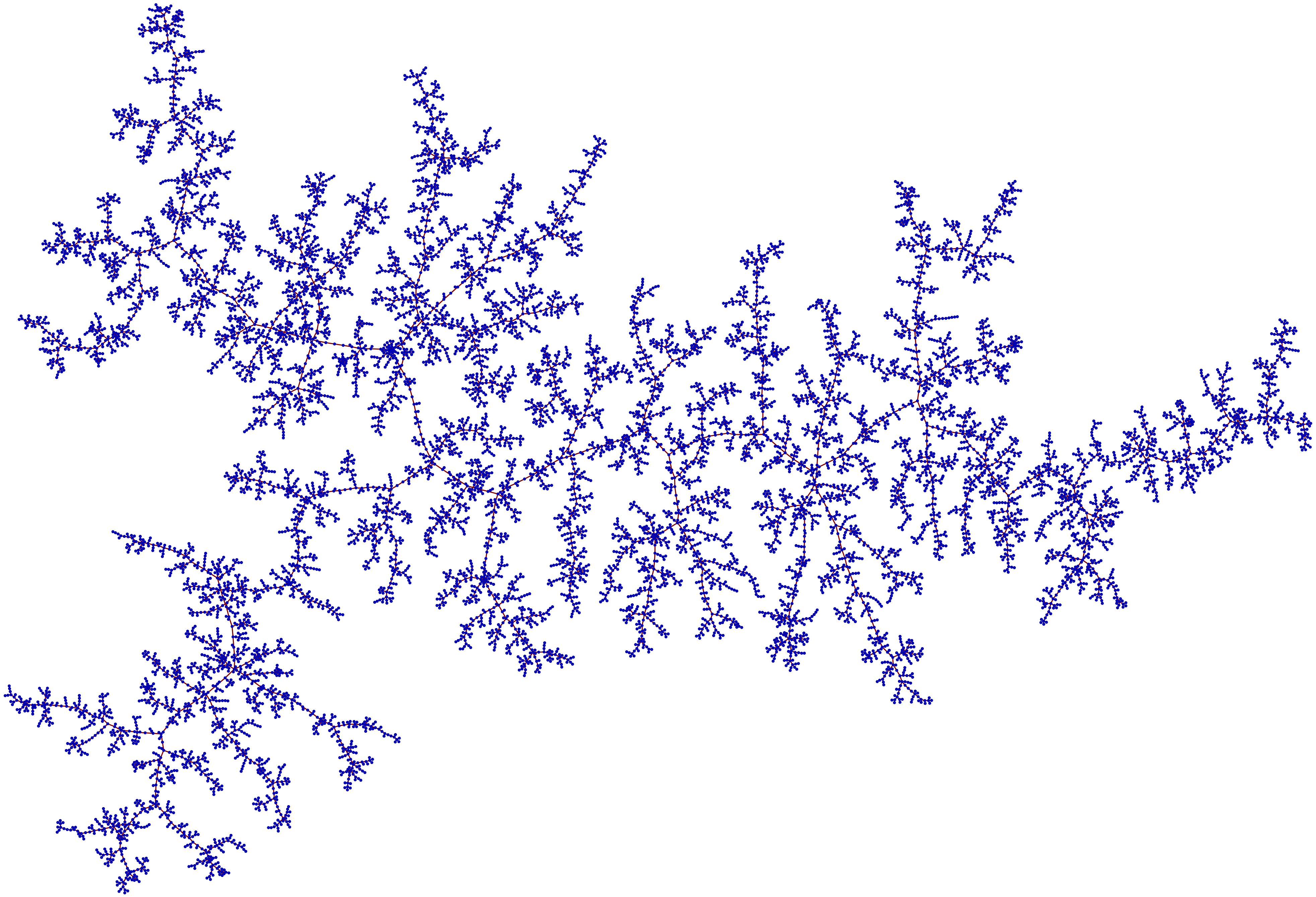
What does t_n look like for large n?

A simulation of a large random critical GW tree



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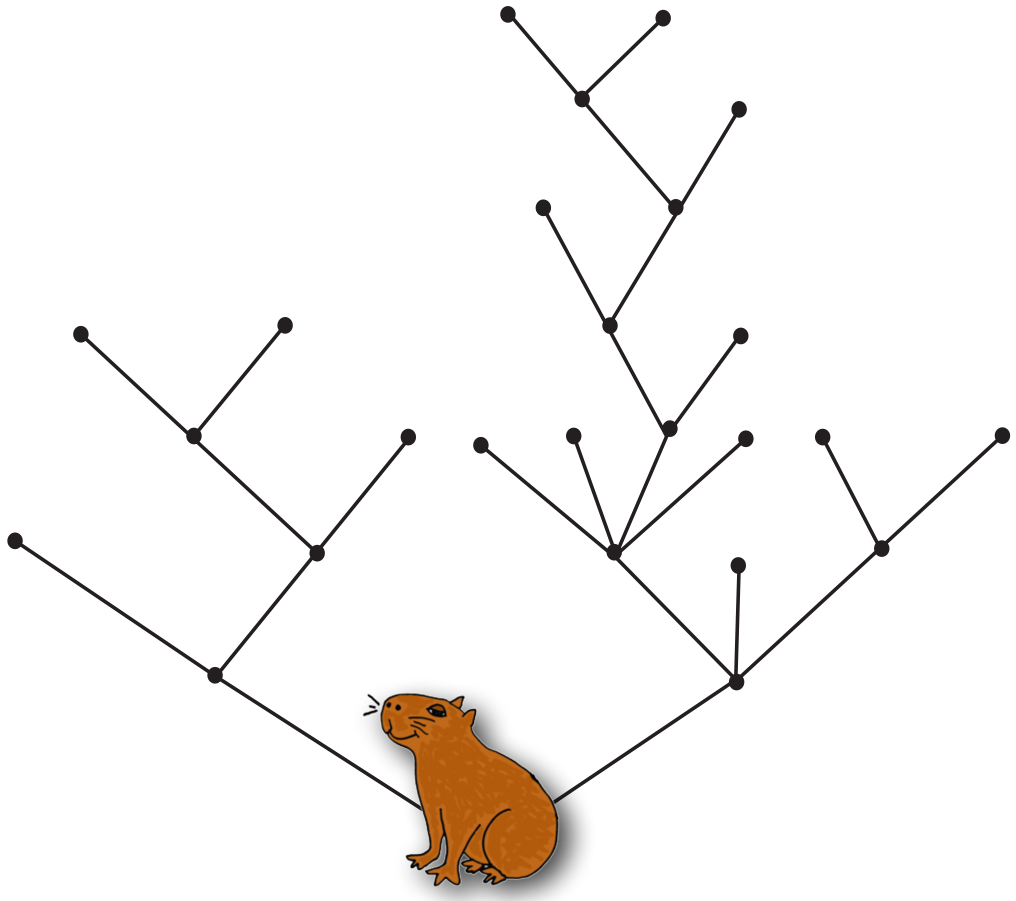
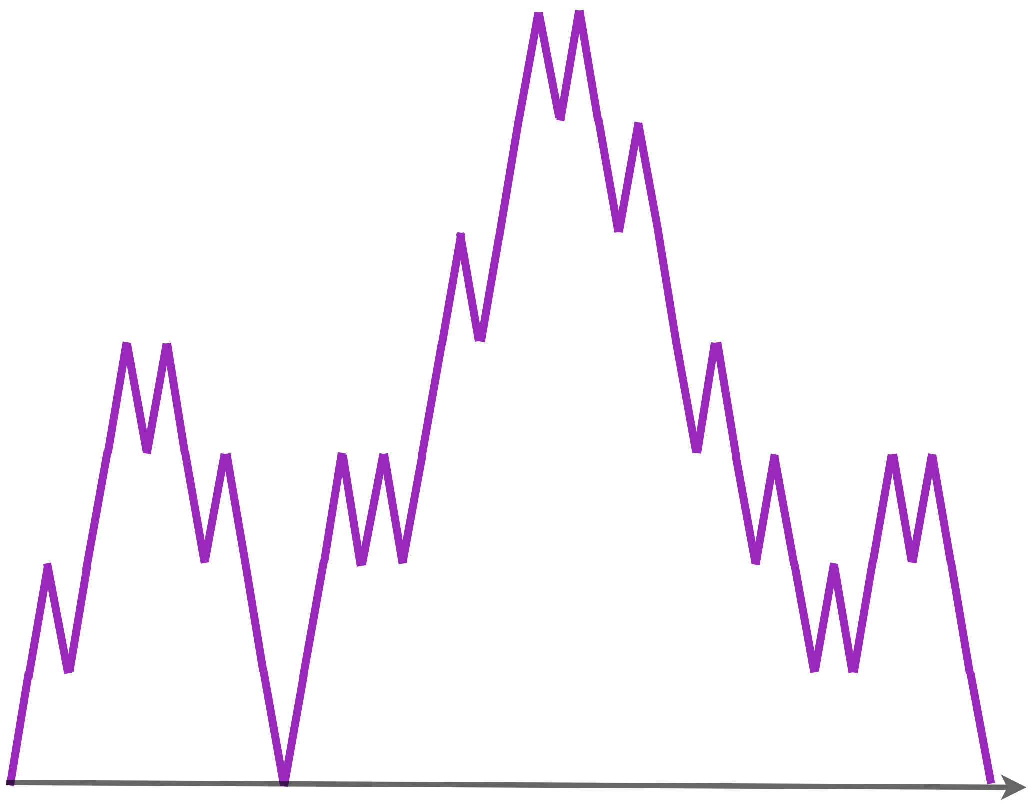
- $\mathcal{T}$ is coded by the normalized Brownian excursion.

What is the Brownian Continuum Random Tree?

First define the contour function of a tree:





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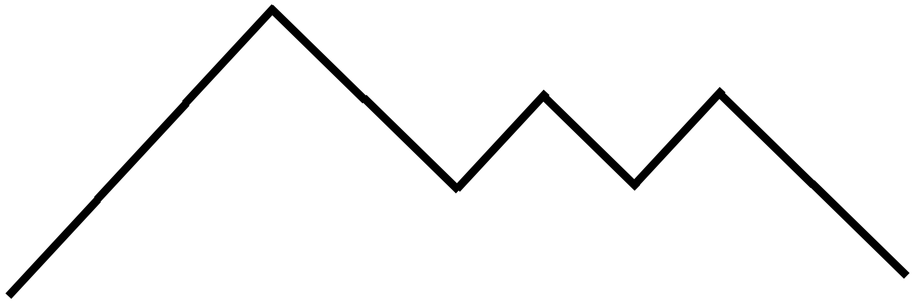
Knowing the contour function, it is easy to recover the tree by gluing:



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The Brownian tree $\mathcal{T}$ is obtained by gluing from the Brownian excursion $\mathbf{e}$.

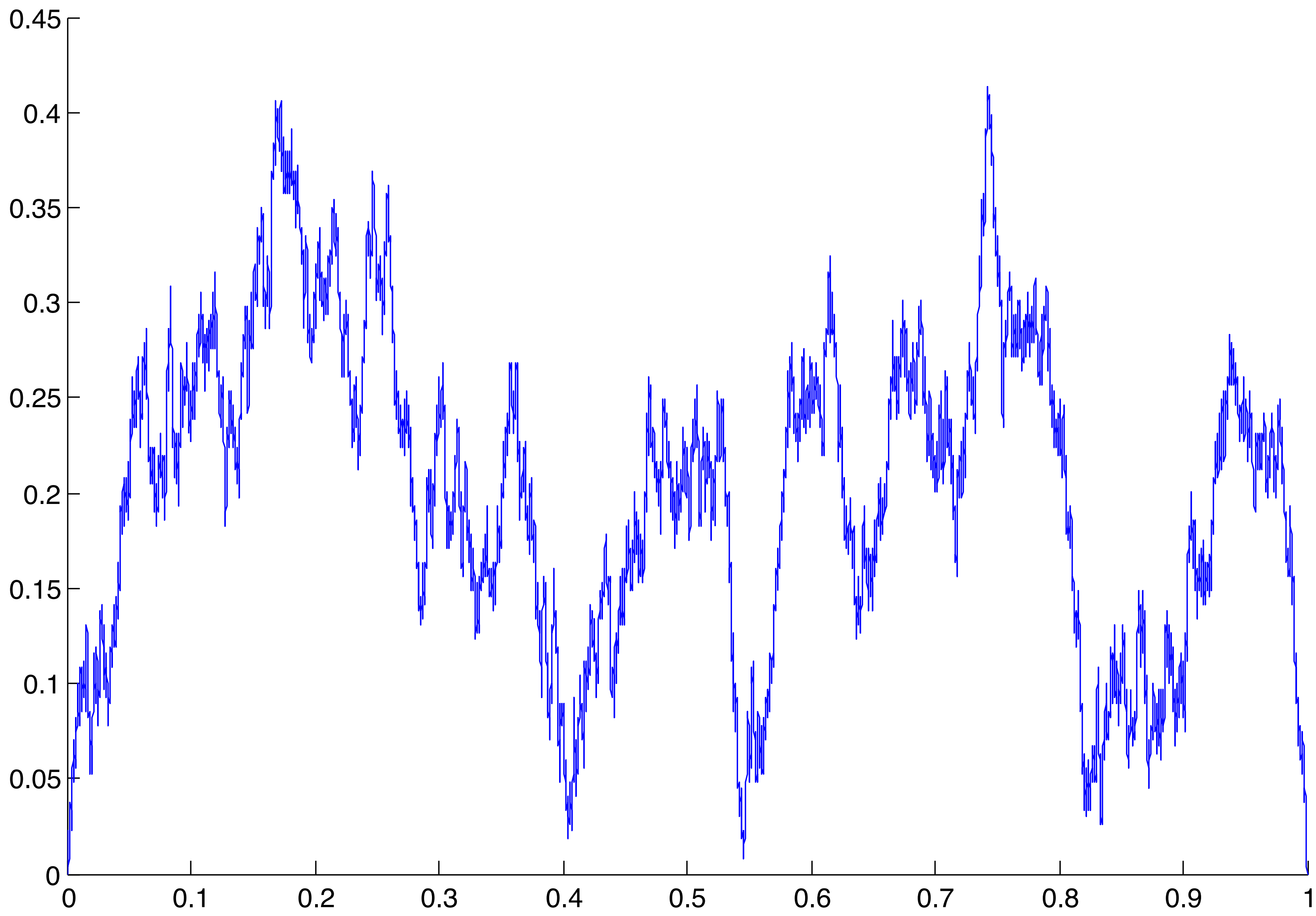

Figure: A simulation of $\mathbf{e}$.

A simulation of the Brownian CRT

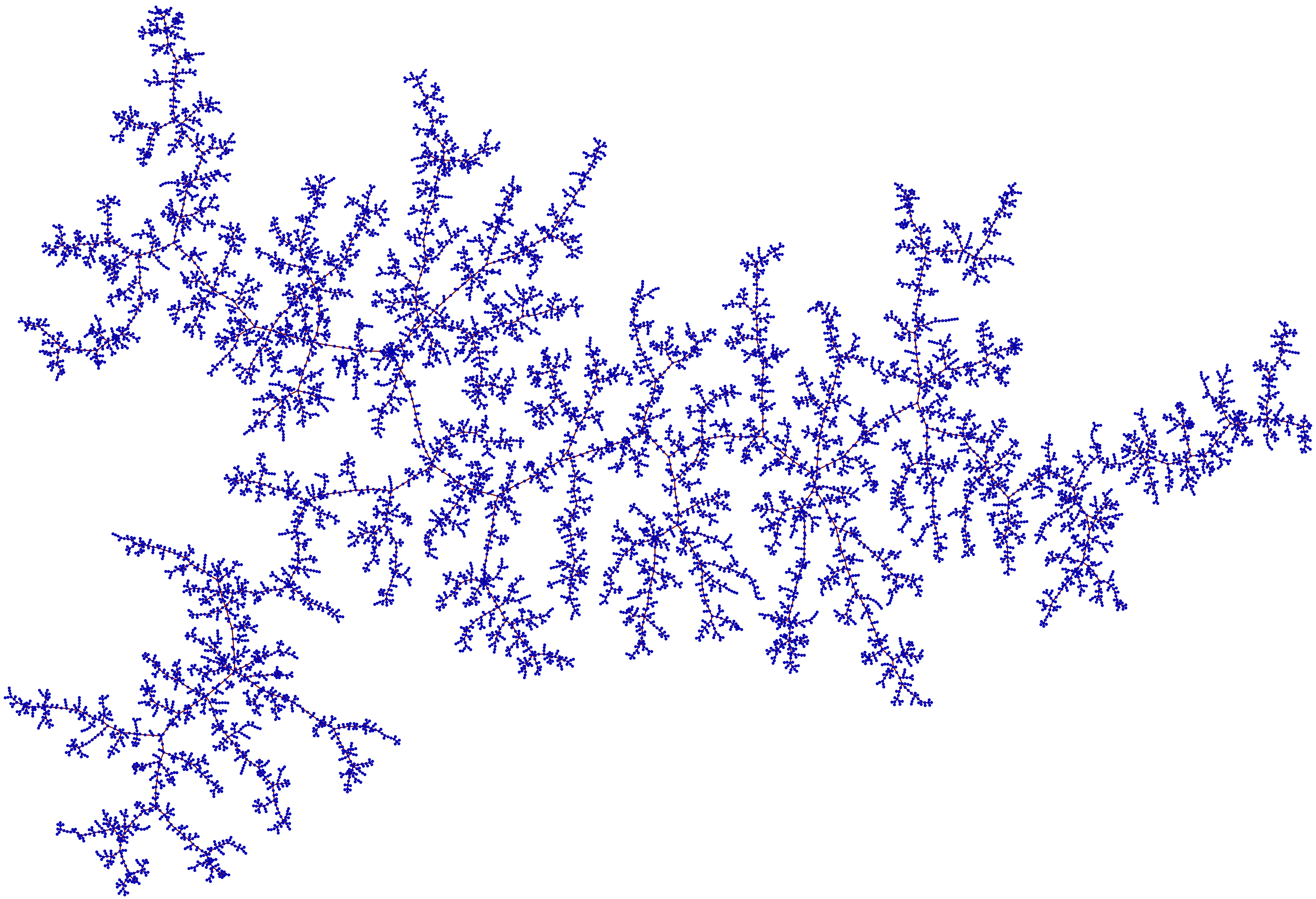

Figure: A non isometric plane embedding of a realization of $\mathcal{T}_{\mathbf{e}}$.

Scaling limits: infinite variance case

Scaling limits: domain of attraction of a stable law

Fix $\alpha \in (1,2)$. Let μ be an offspring distribution such that

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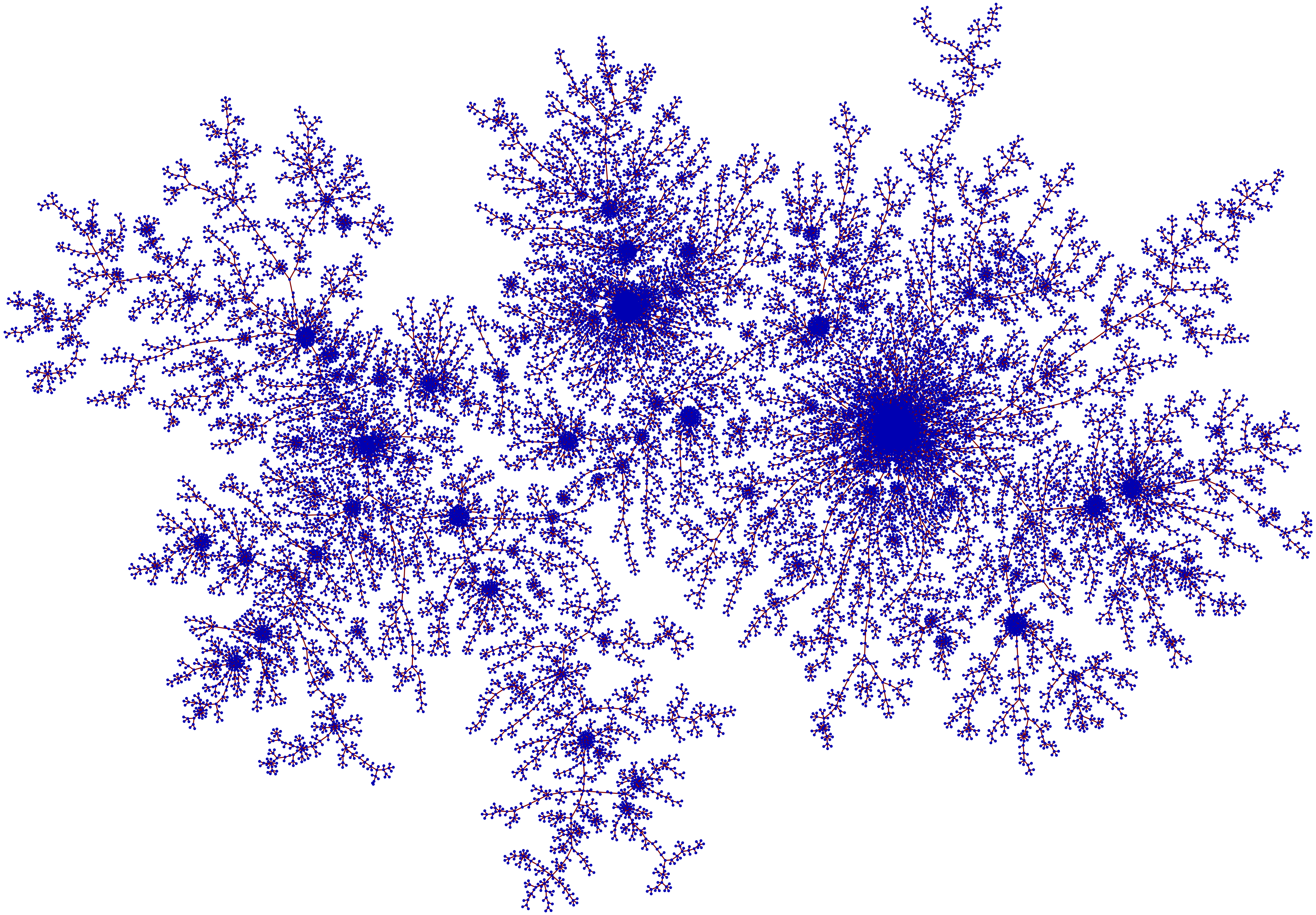

Figure: A large $\alpha = 1.1$ – stable tree

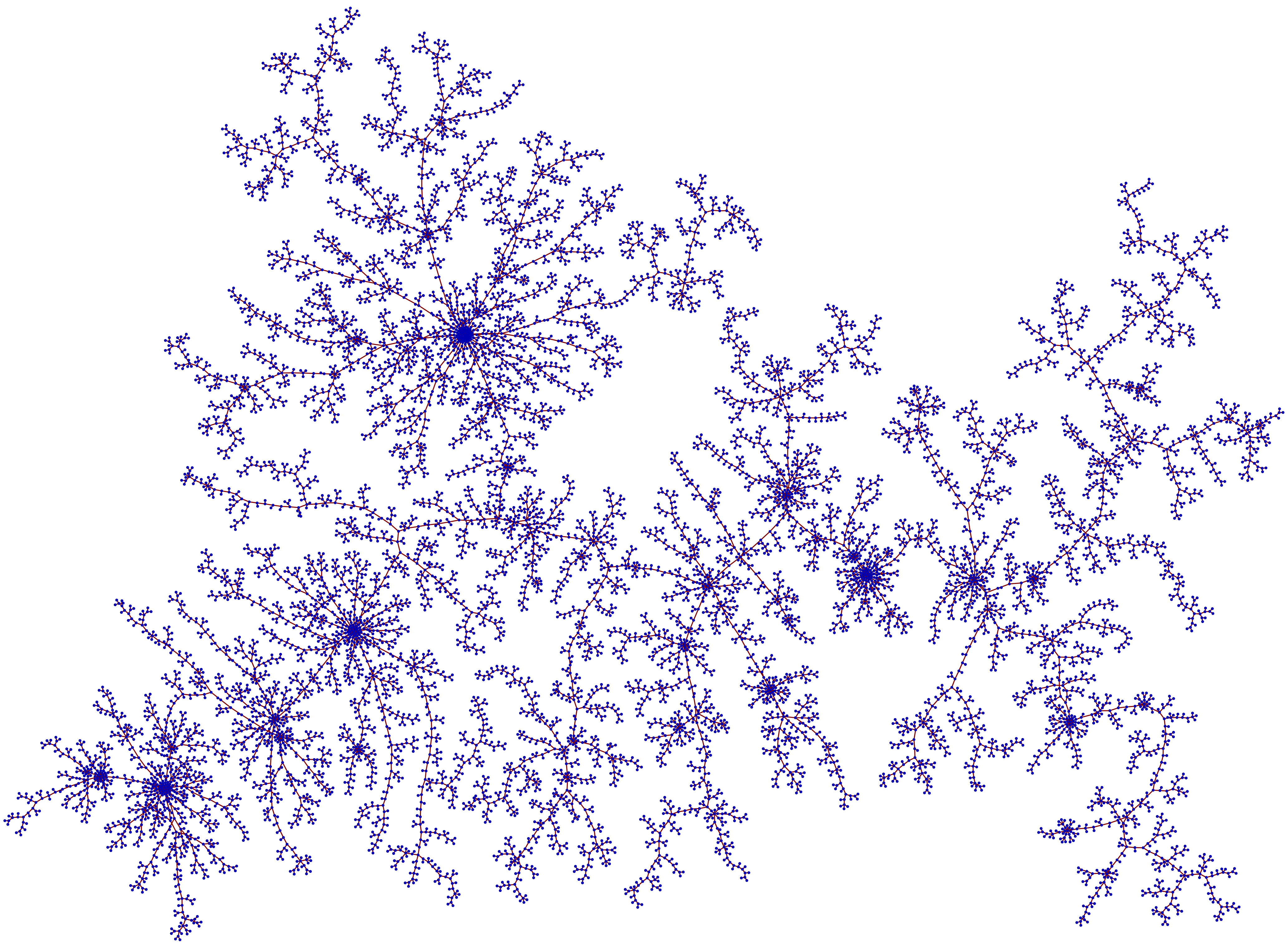

Figure: A large $\alpha = 1.5$ – stable tree

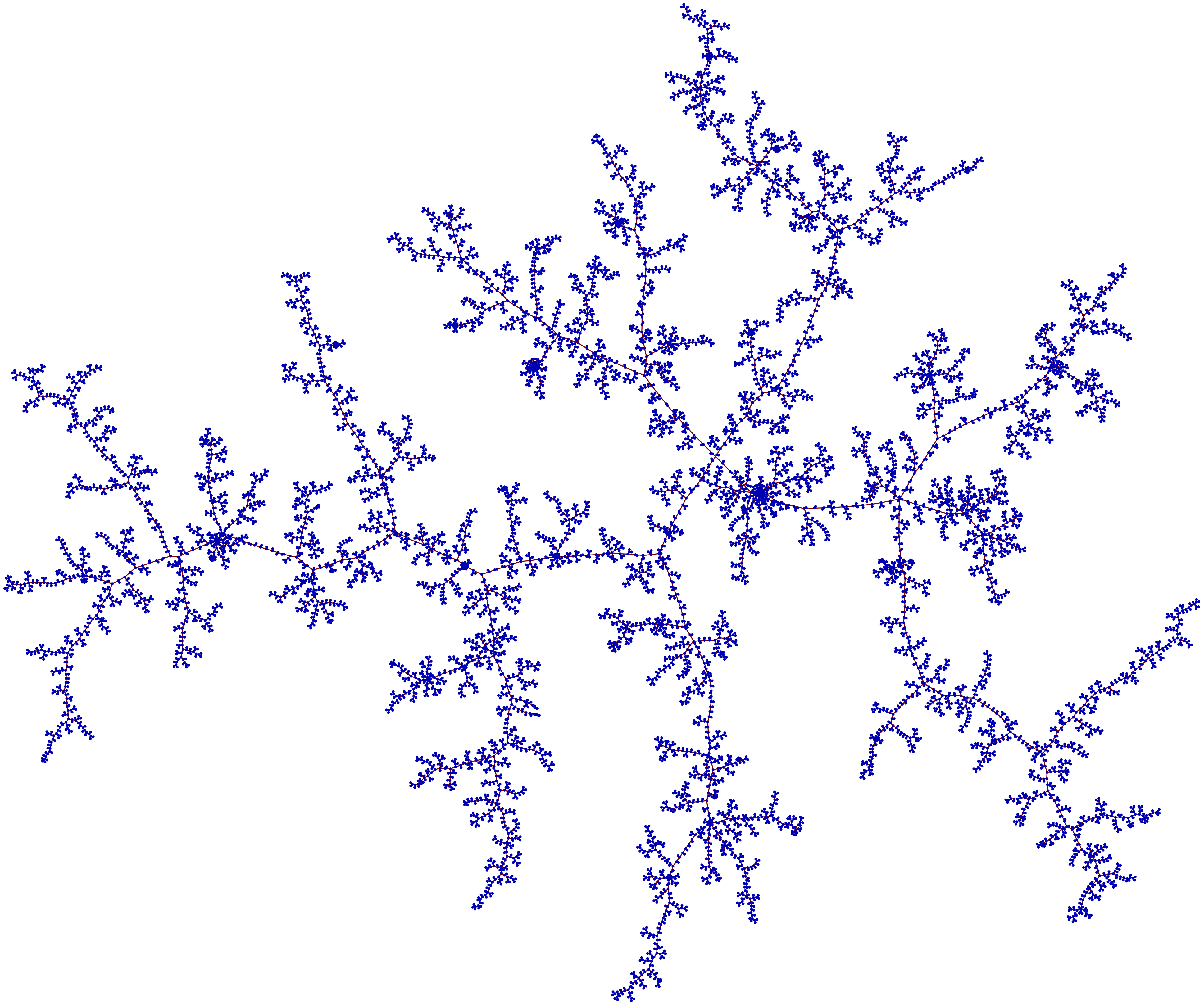

Figure: A large $\alpha = 1.9$ – stable tree

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Theorem (Duquesne '03)

There exists a random compact metric space $\mathcal{T}_\alpha$ such that:

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The tree $\mathcal{T}_\alpha$ is called the stable tree of index α (introduced by Le Gall & Le Jan).

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- The maximal degree of t_n is of order $n^{1/\alpha}$.
- $\mathcal{T}_\alpha$ is coded by the normalized excursion of a spectrally positive stable Lévy process of index α.

What happens if μ is not critical?

Scaling limits: nongeneric case

Fix $\alpha > 1$. Let μ be a **subcritical** offspring distribution such that $\mu_i \sim c/i^{1+\alpha}$.

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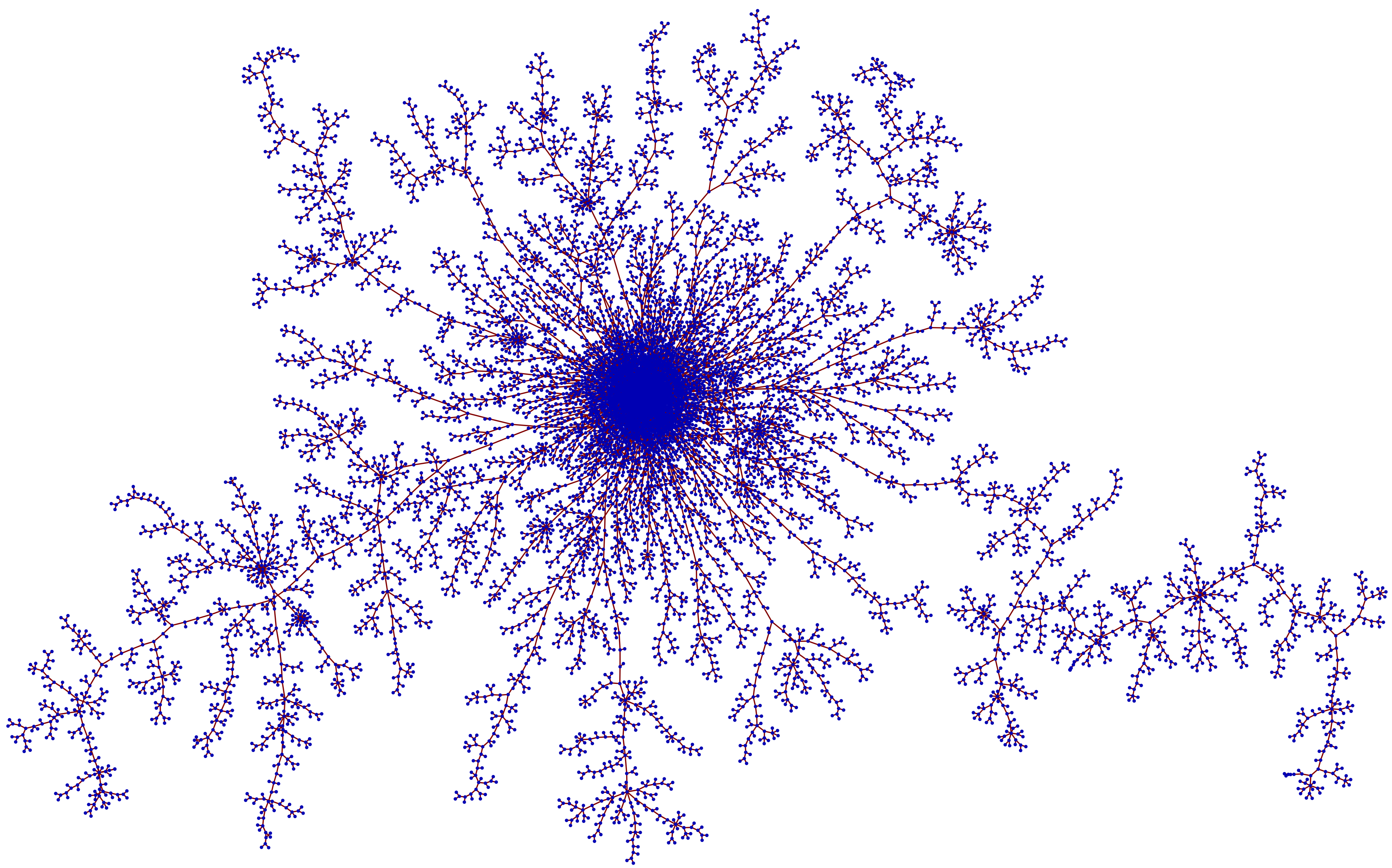

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Theorem (K.).

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III. Looptrees and preferential attachment

IV. Looptrees and percolation on random triangulations

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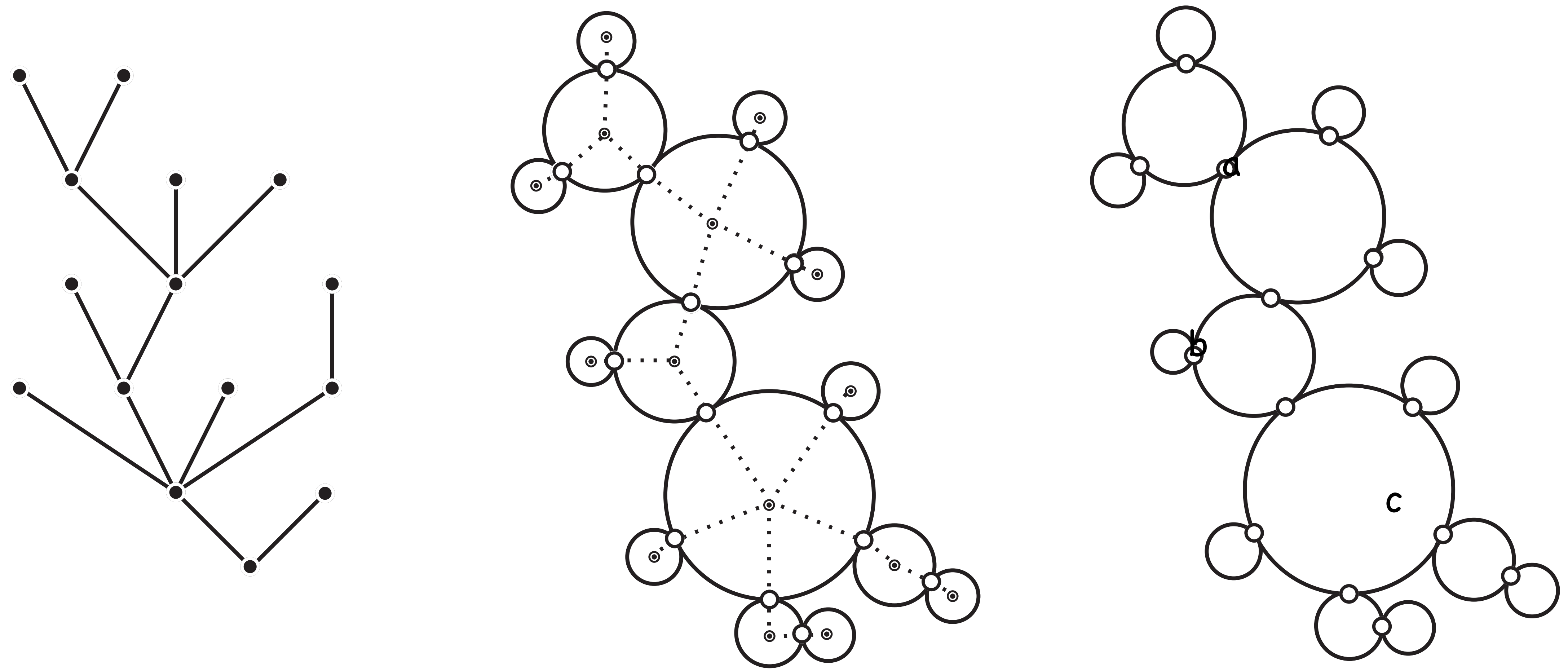

Figure: A plane tree τ and its associated discrete looptree $\mathrm{Loop}(\tau)$.

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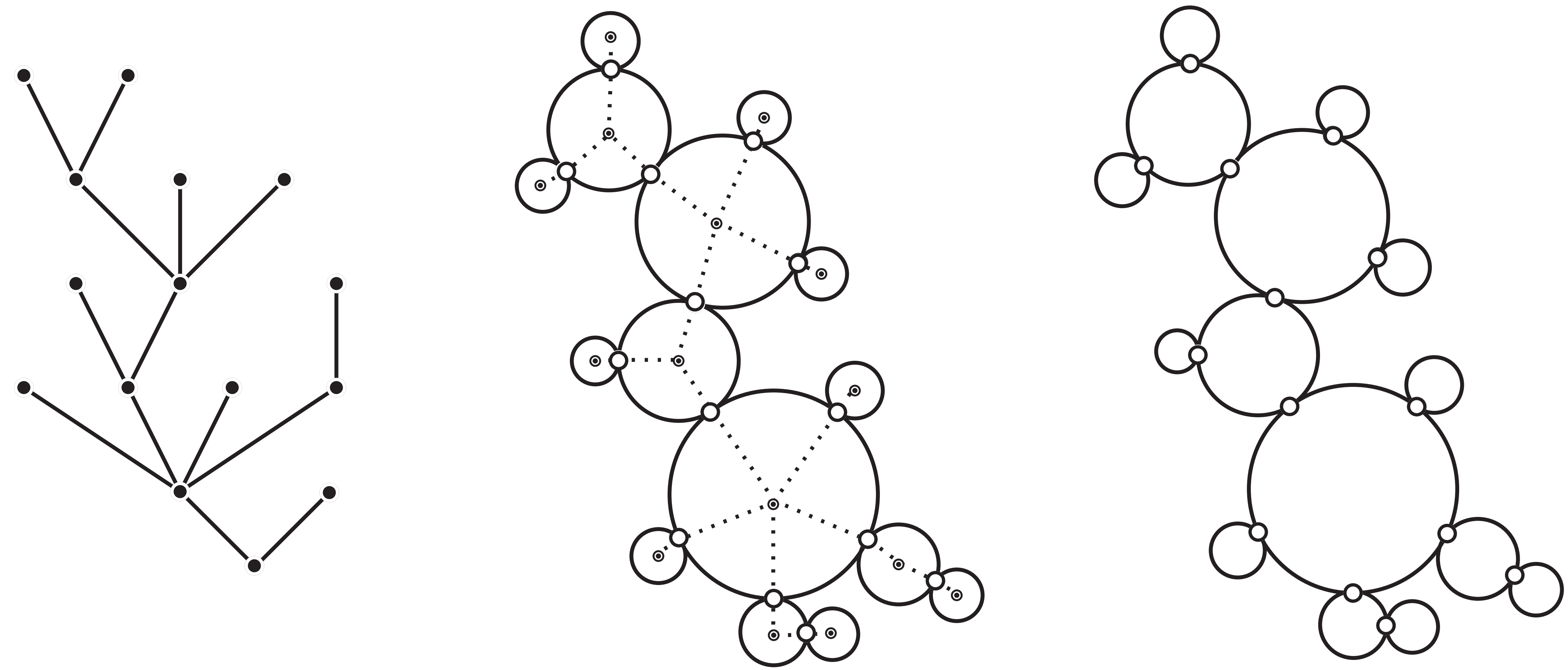

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We view $\mathrm{Loop}(\tau)$ as a compact metric space.

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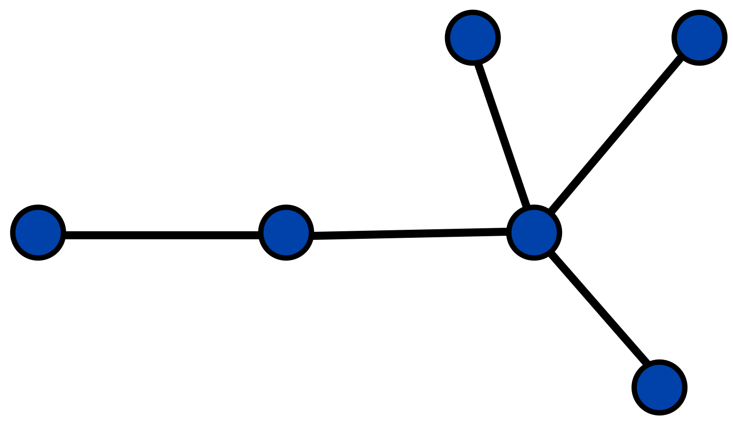

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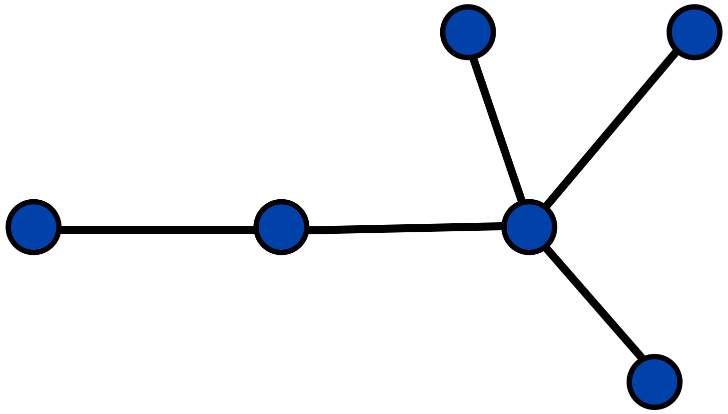

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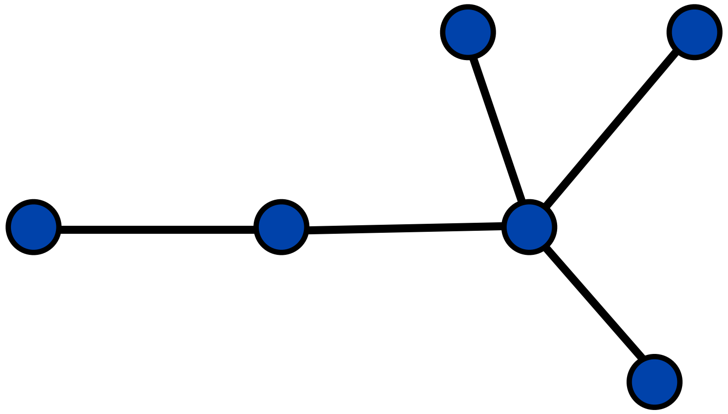

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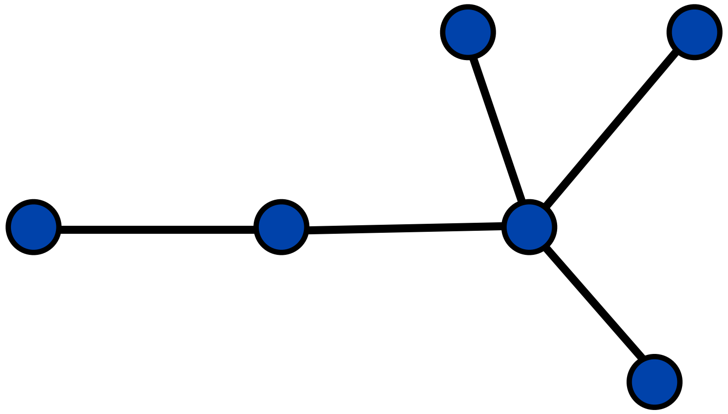

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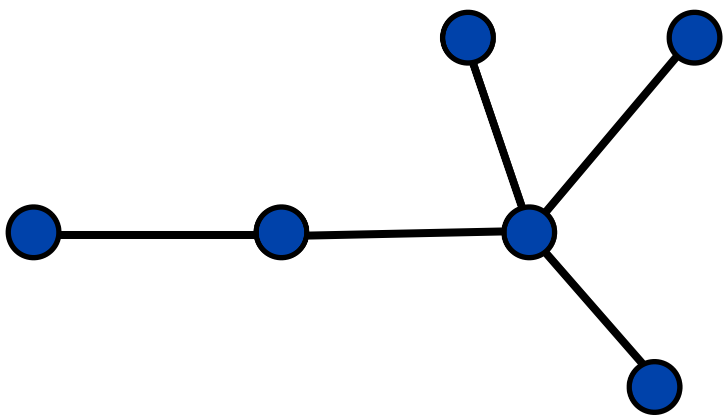

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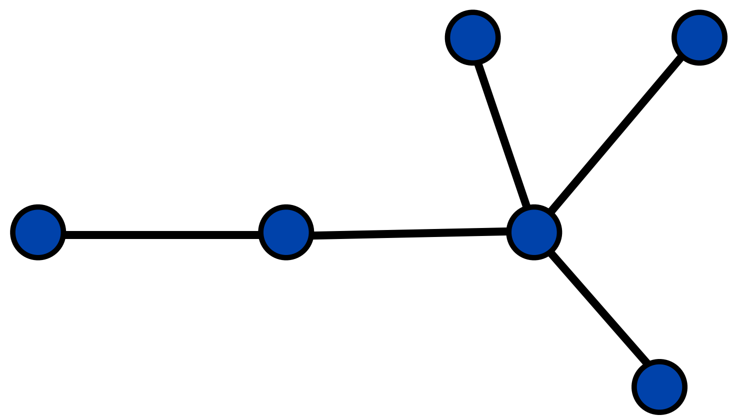

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⇝ Answer: no.

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Theorem (Curien, Duquesne, K., Manolescu)**.**

There exists a random compact metric space $\mathcal{L}$ such that:

$$n^{-1/2} \cdot \mathrm{Loop}(T_n) \quad \xrightarrow[n\to\infty]{a.s.} \quad \mathcal{L},$$

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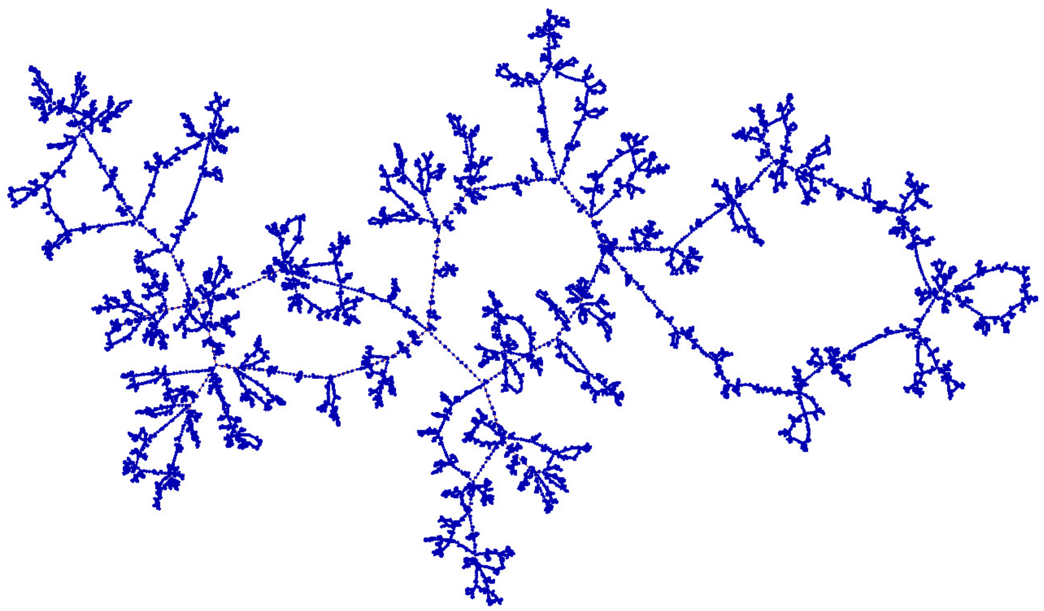

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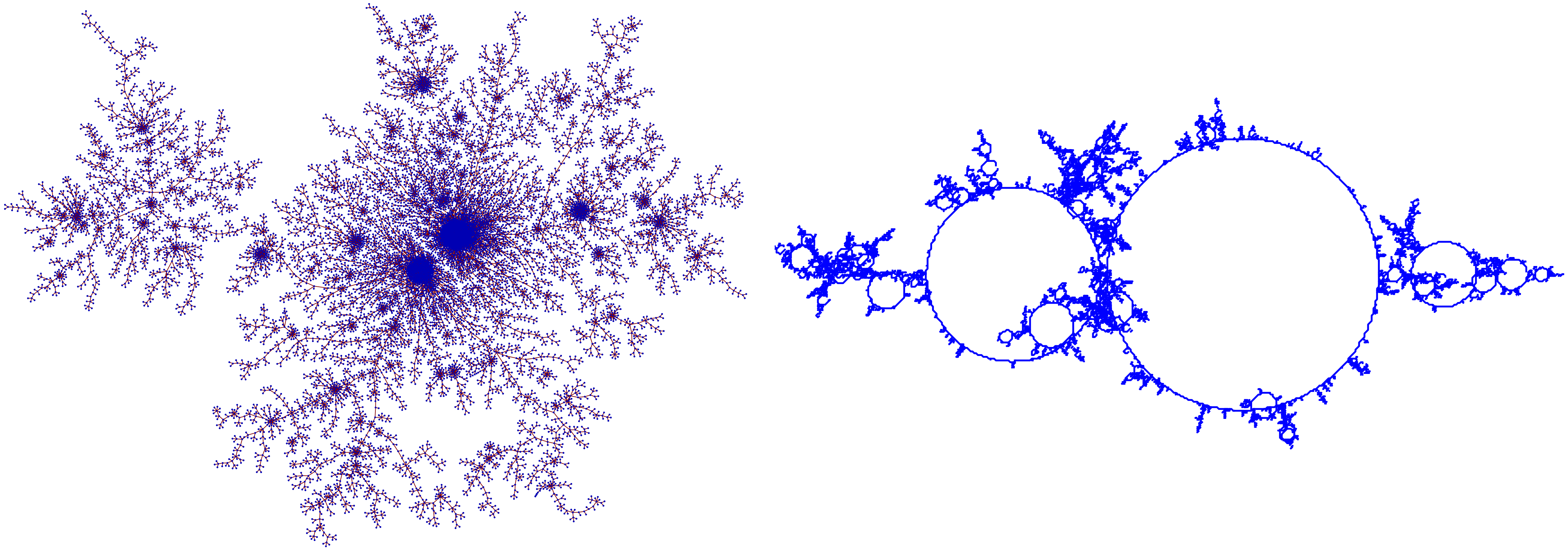

Figure: A large $\alpha = 1.1$ tree and its associated looptree.

Triangulations

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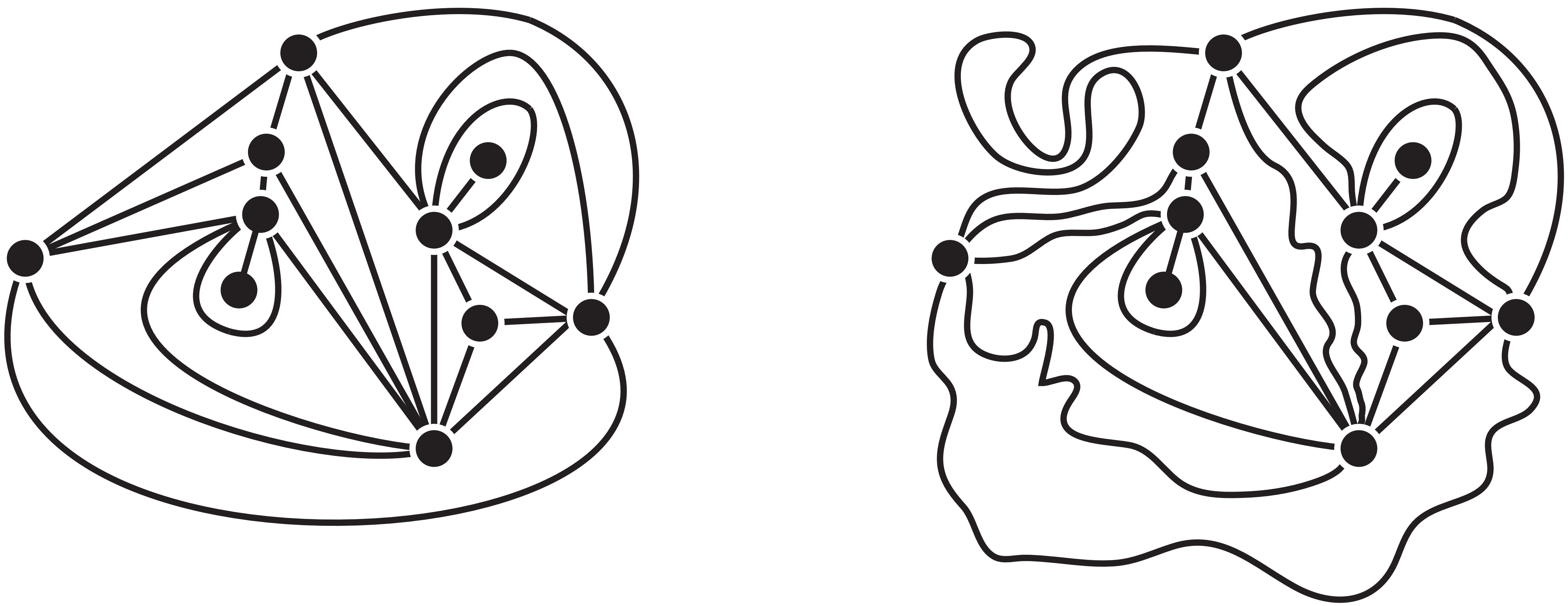

Figure: Two identical triangulations.

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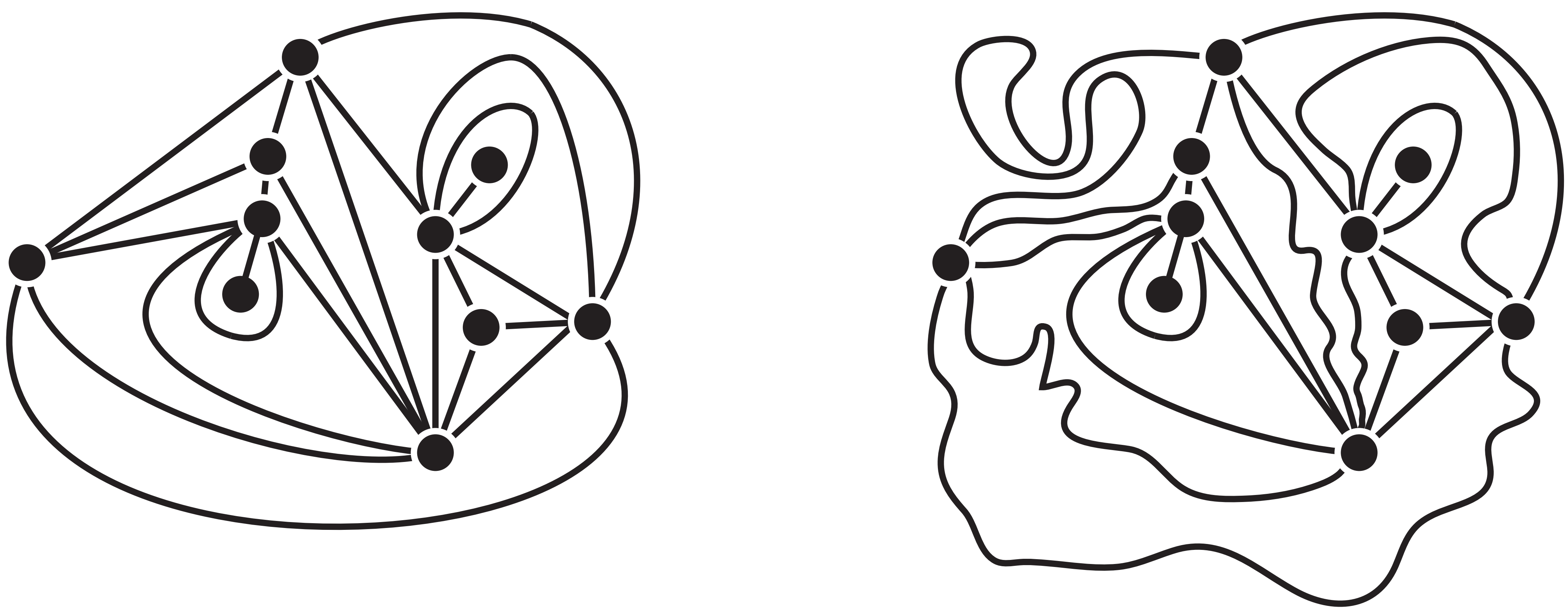

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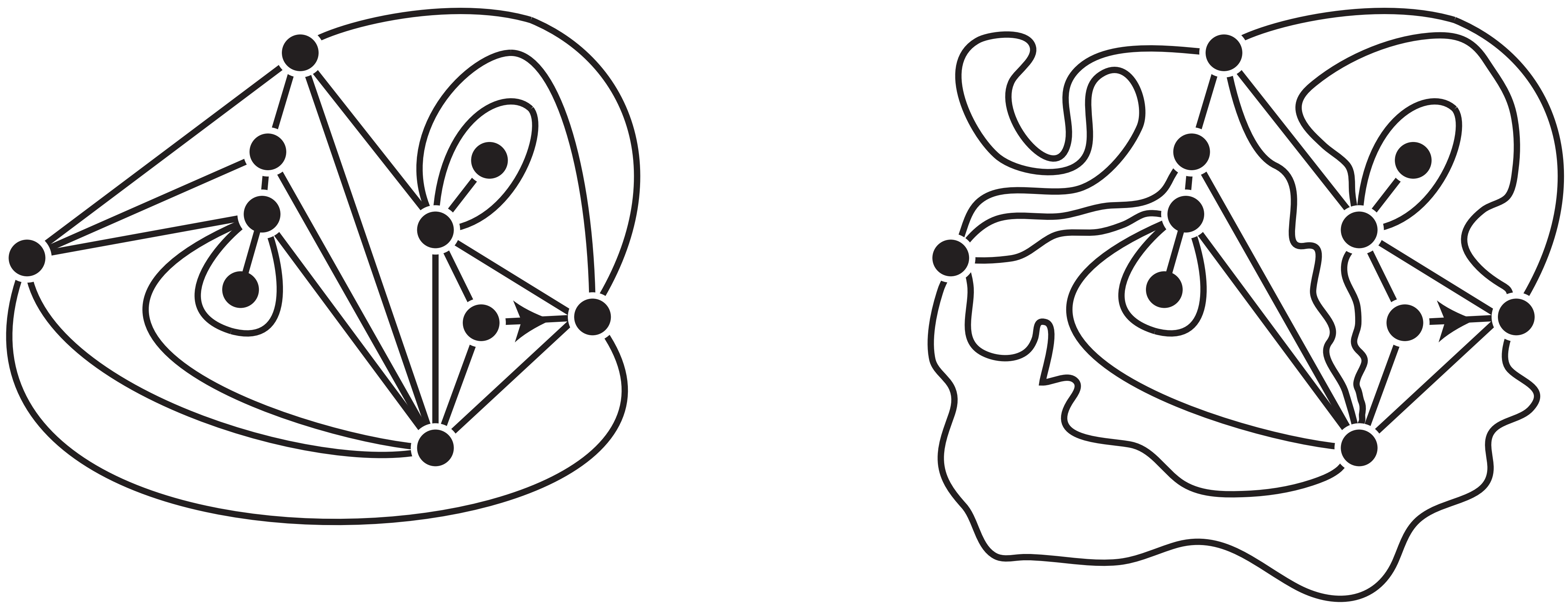

Figure: Two identical rooted triangulations.

The UIPT

Angel & Schramm defined an infinite triangulation T_∞, called the Uniform Infinite Plane Triangulation (UIPT), by local approximations using finite uniform random triangulations.

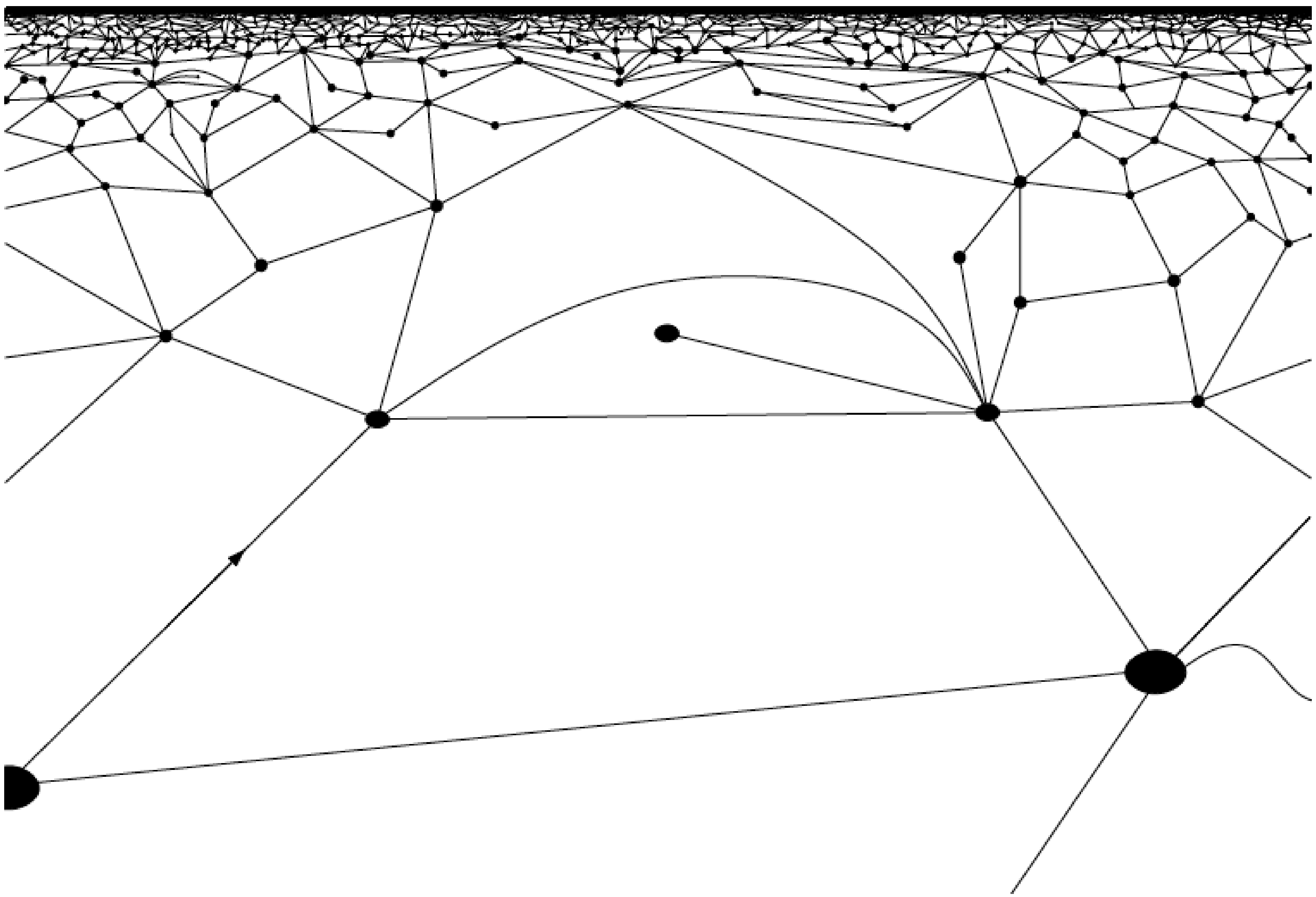

Figure: (due to N. Curien): an artistic view of the UIPQ.

Percolation on the UIPT

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Conditionally on T_∞, consider a site percolation with parameter $p \in (0, 1)$

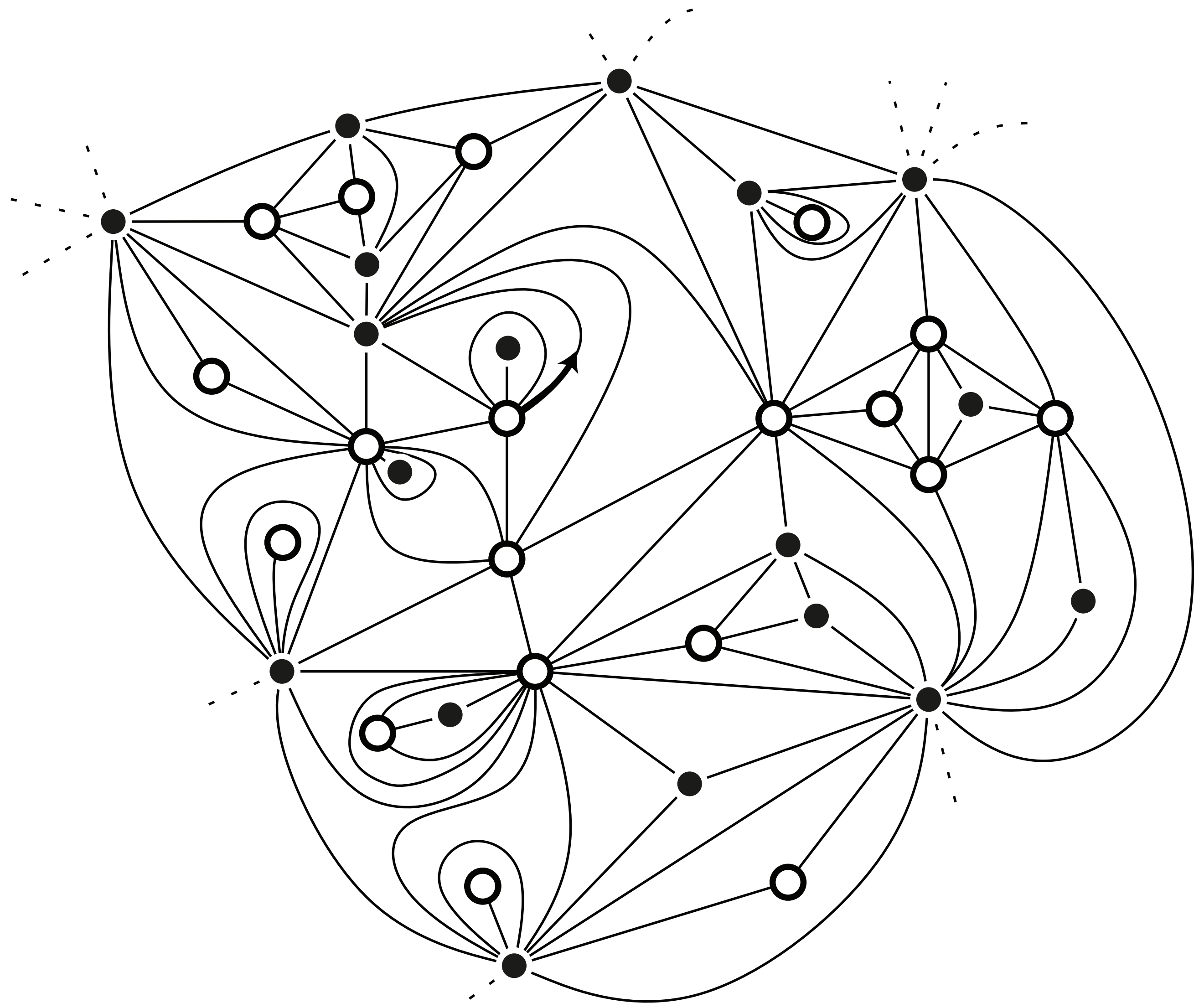

Figure: A realization of a site percolation on the UIPT.

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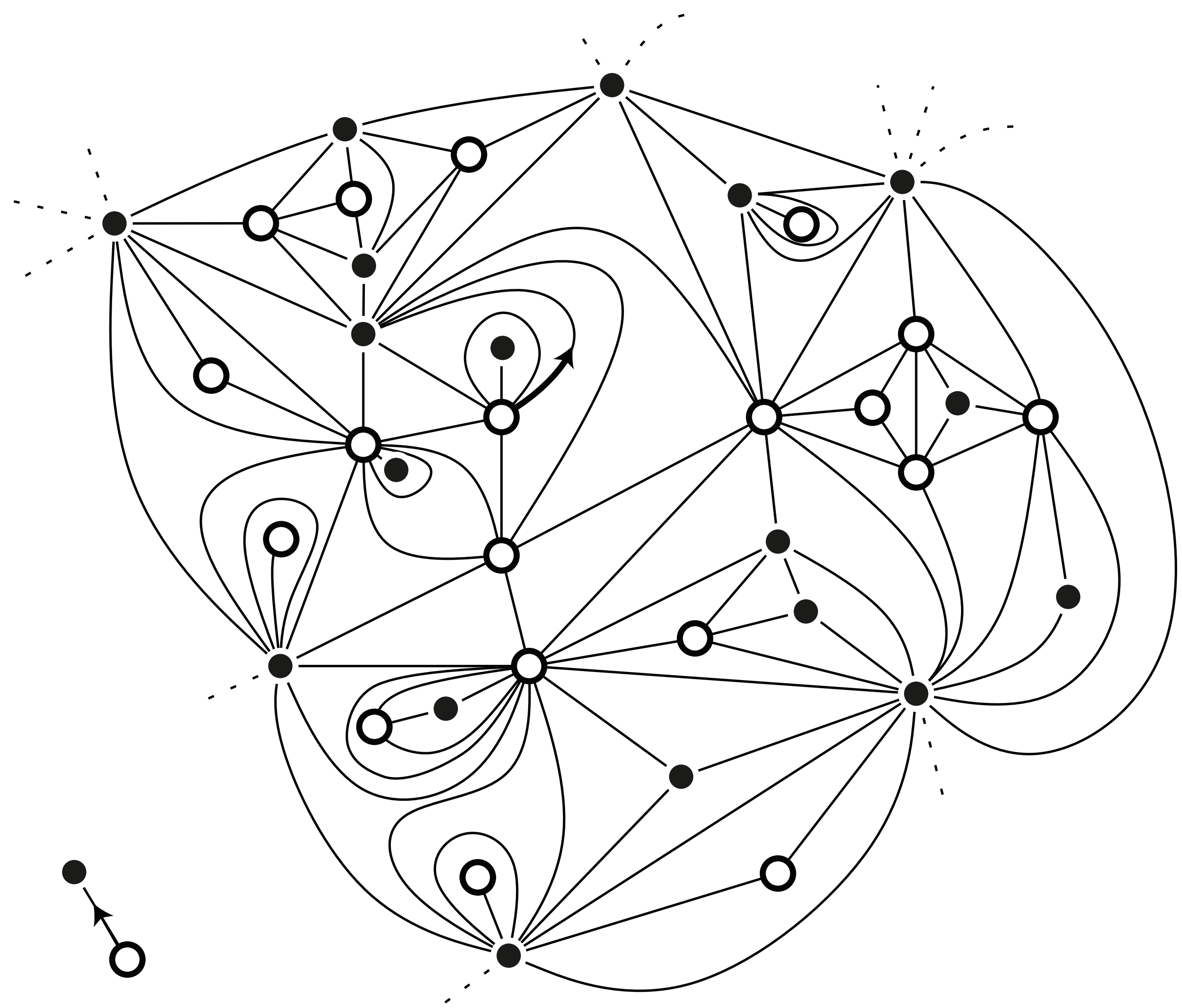

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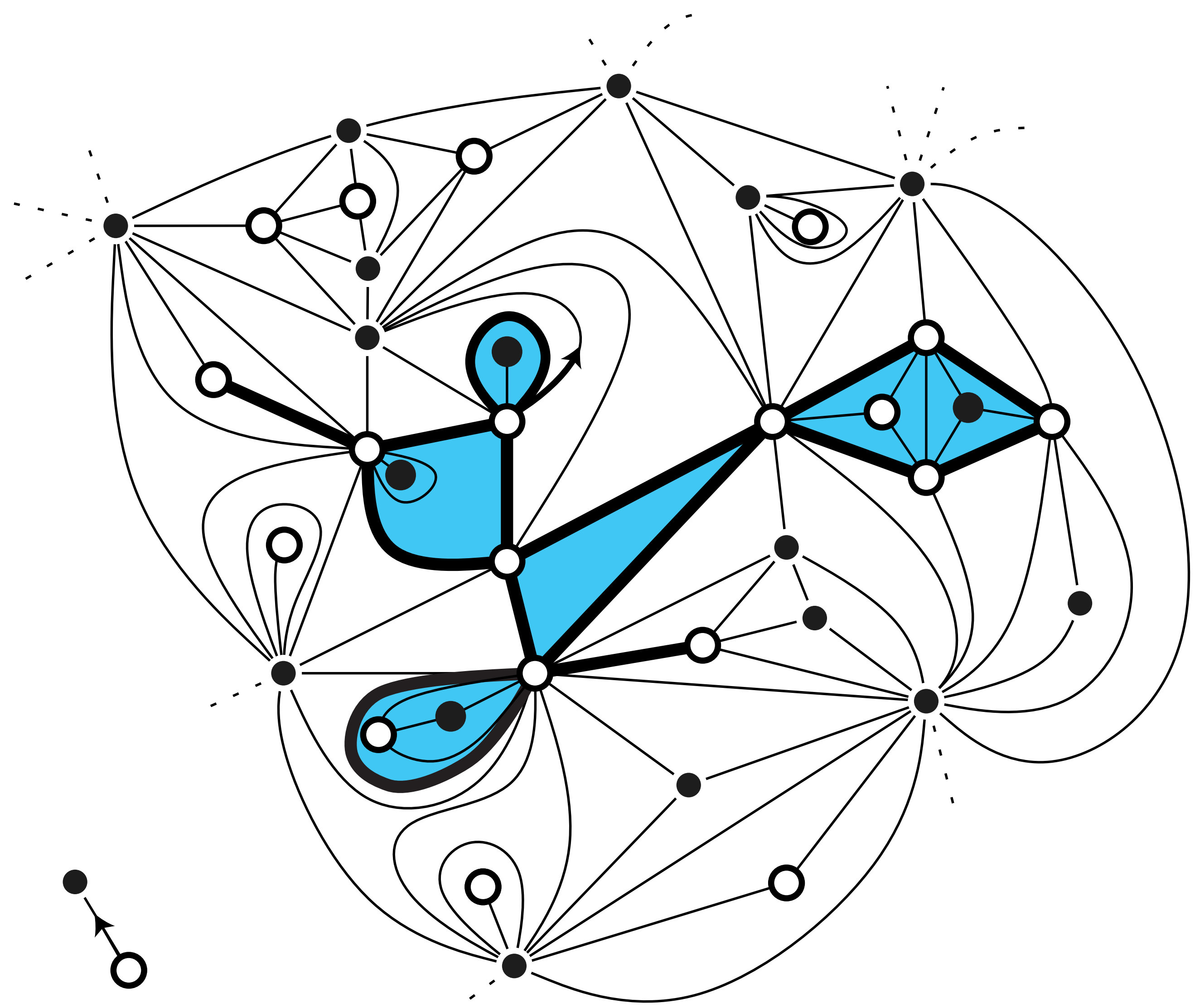

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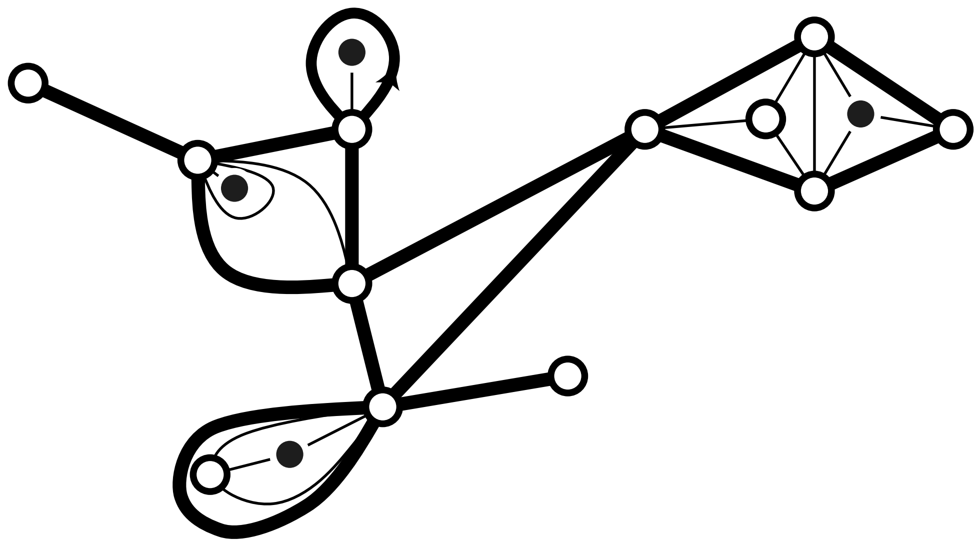

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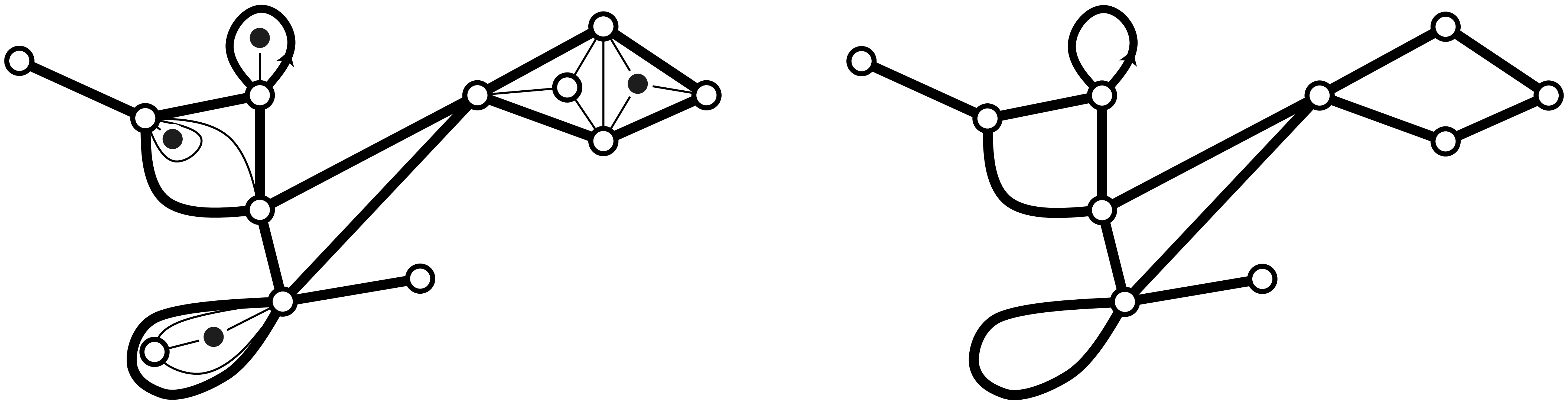

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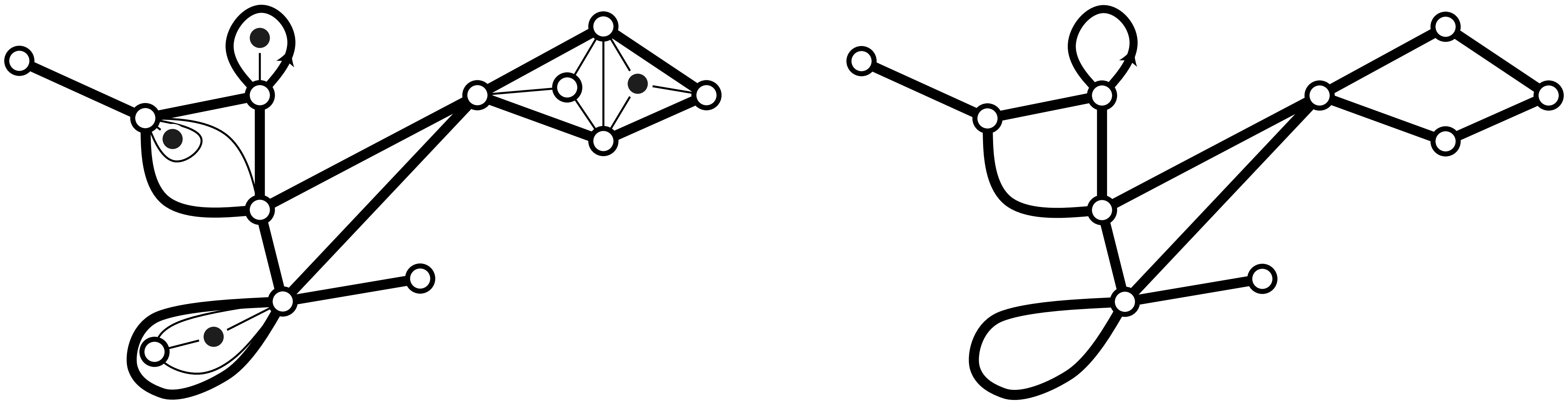

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By definition, $\#\partial\mathcal{H}$ is the number of **half-edges** on $\partial\mathcal{H}$.

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Here, $\mathcal{T}$ is the Brownian CRT and $\mathcal{C}_1$ is the circle of unit length.

Fix $p \in (0,1)$, and for $m \geqslant 1$ let $\partial \mathcal{H}_m^{(p)}$ be the random variable $\partial \mathcal{H}$ conditioned by the event $\{\#\partial \mathcal{H} = m\}$.

Theorem (Curien & K.).

When $0 < p < 1/2$, $\quad m^{-1/2} \cdot \partial \mathcal{H}_m^{(p)} \xrightarrow[m \to \infty]{(d)} C_p \cdot \mathcal{T}$,

when $p = 1/2$, $\quad m^{-2/3} \cdot \partial \mathcal{H}_m^{(1/2)} \xrightarrow[m \to \infty]{(d)} 3^{1/3} \cdot \mathscr{L}_{3/2}$,

when $1/2 < p < 1$, $\quad m^{-1} \cdot \partial \mathcal{H}_m^{(p)} \xrightarrow[m \to \infty]{(d)} \frac{2p-1}{\sqrt{3}-1+2p} \cdot \mathcal{C}_1$.

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Remark

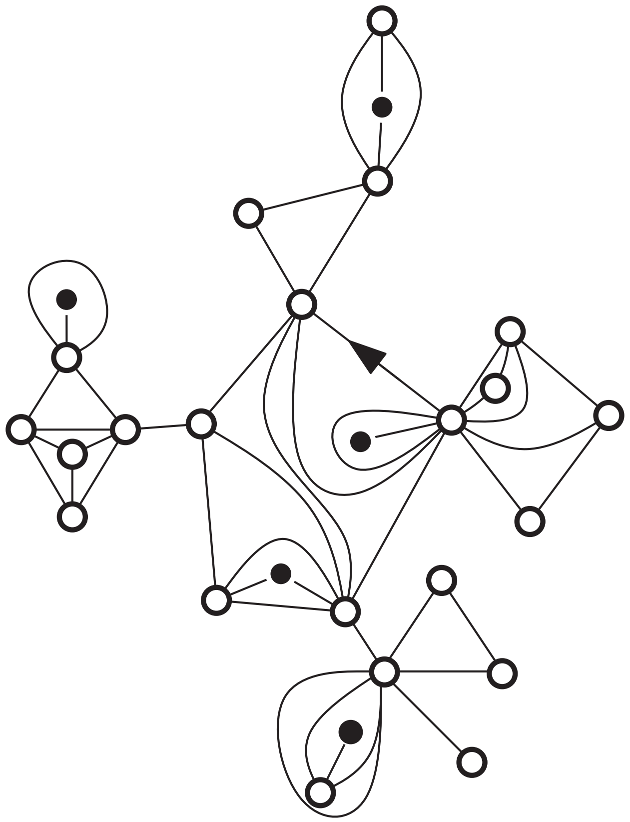
- Angel proved that $p_c = 1/2$.

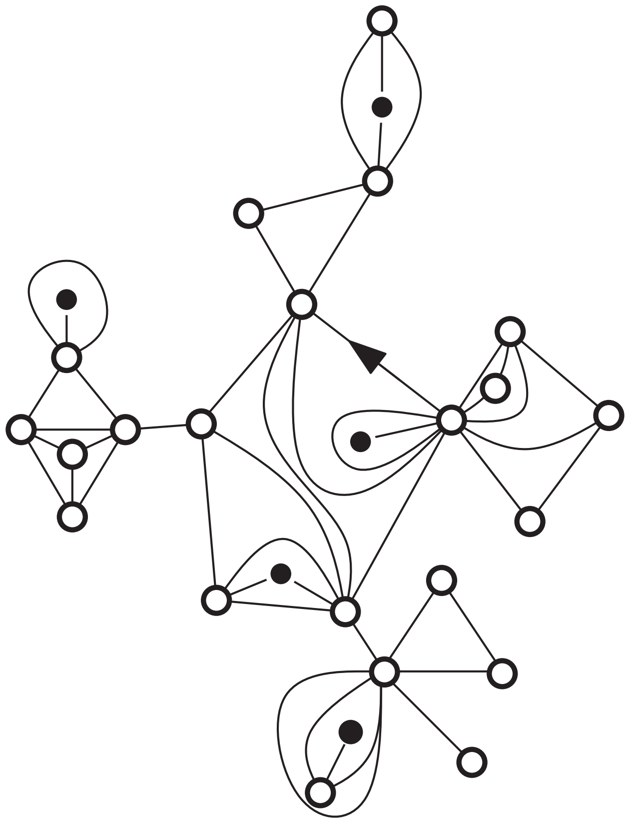
Theorem (Curien & K.)**.**

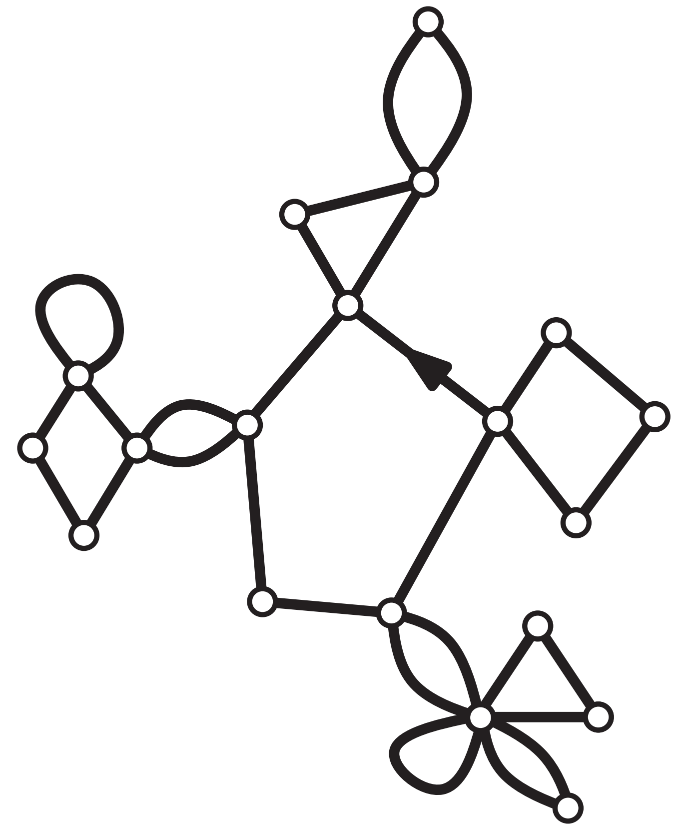
We have:

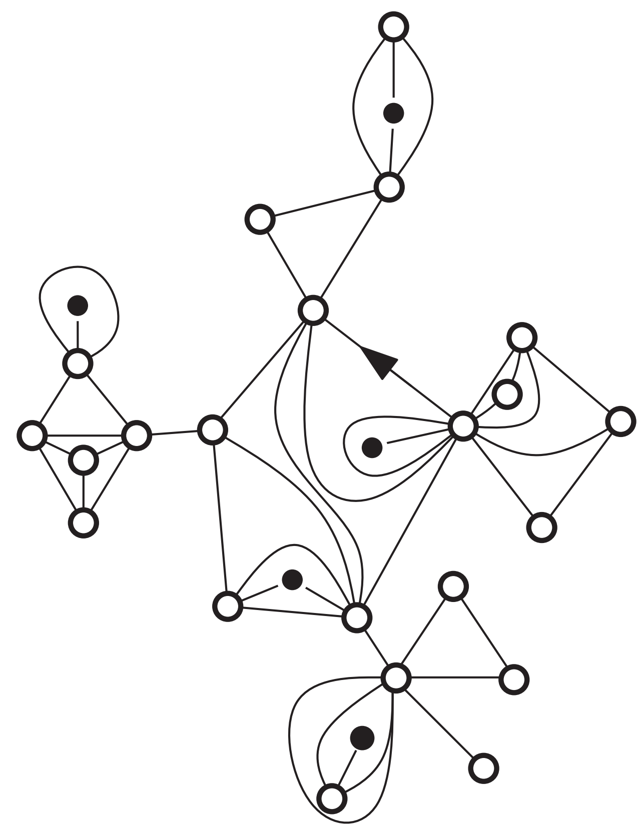
$$\mathbb{P}(\#\partial\mathcal{H}^{(1/2)} = m) \underset{m\to\infty}{\sim} \frac{3}{2 \cdot |\Gamma(-2/3)|^3} \cdot m^{-4/3}.$$

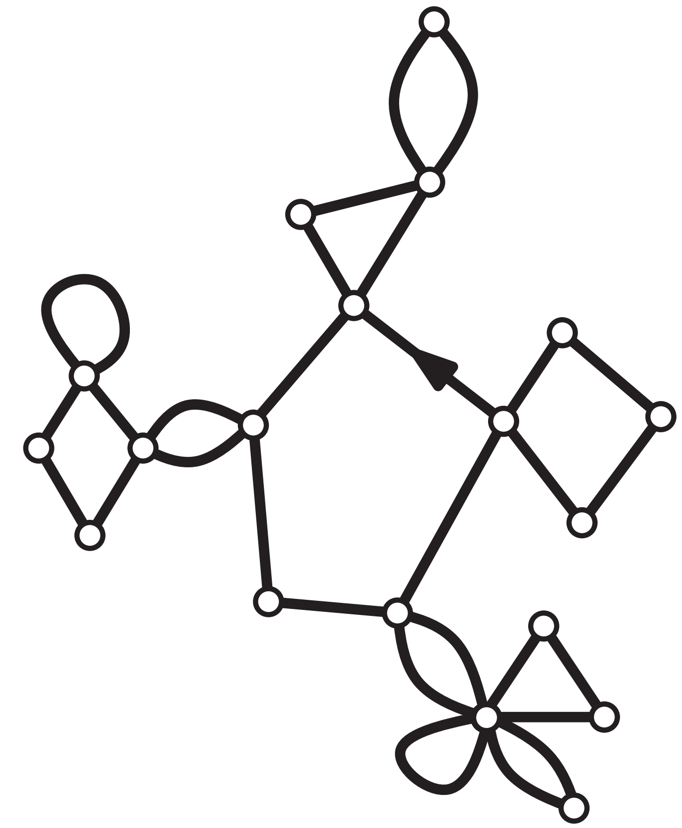
Idea of the proof: Galton–Watson trees

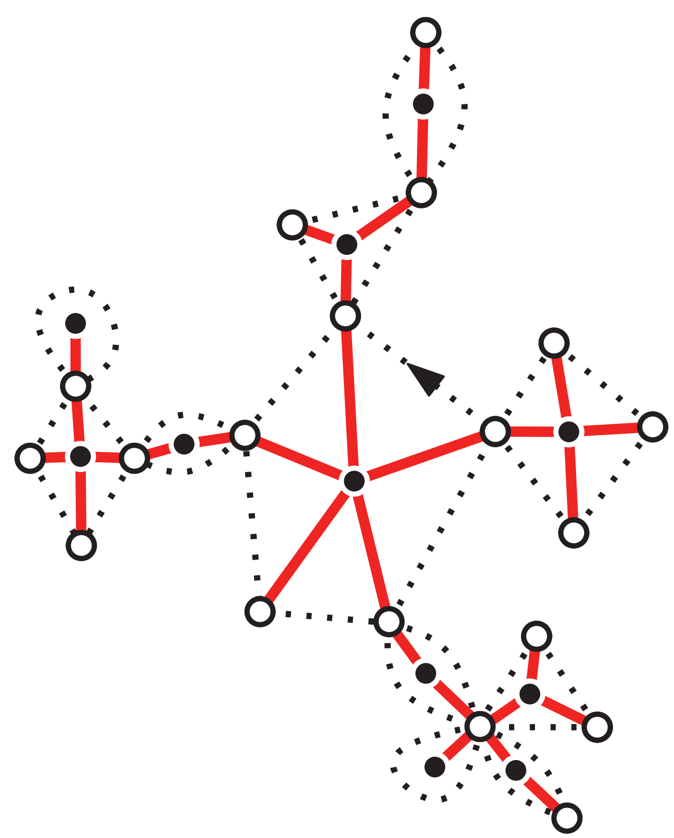


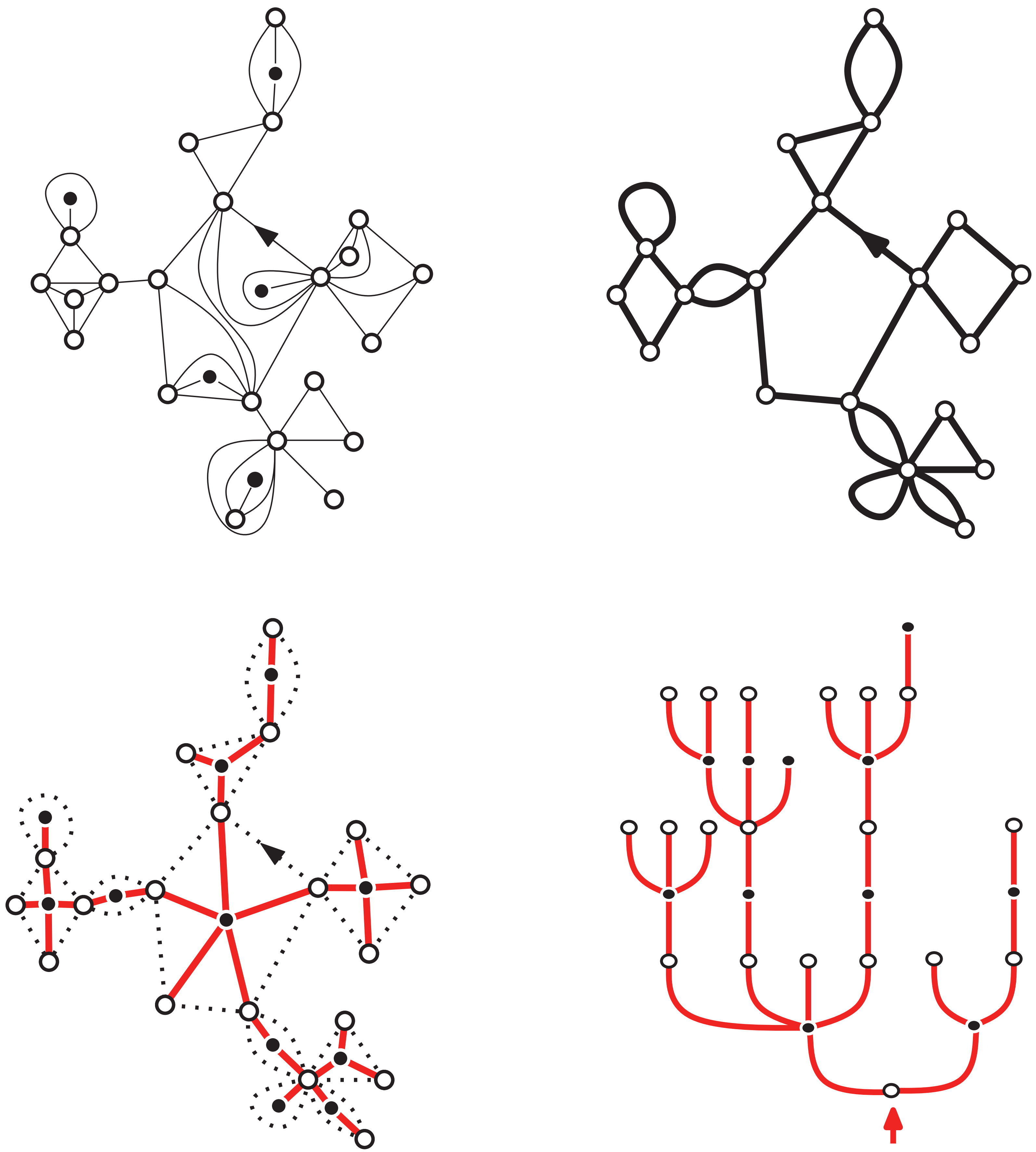












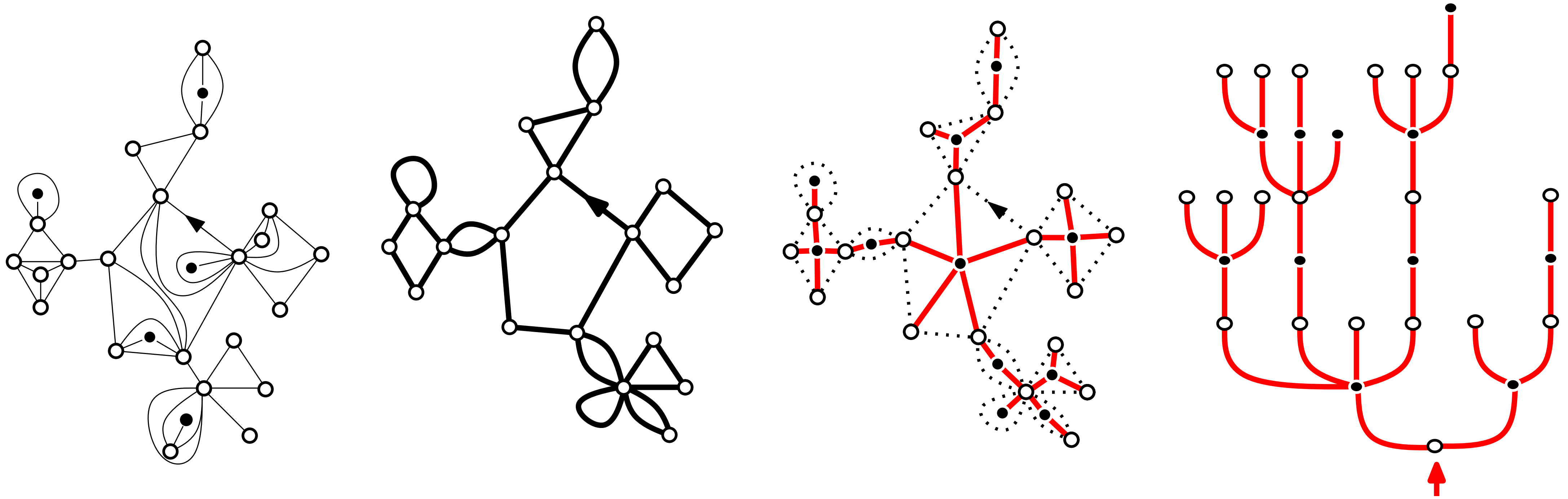


Figure: Construction of a two-type tree $\textbf{Tree}(\partial\mathcal{H})$.

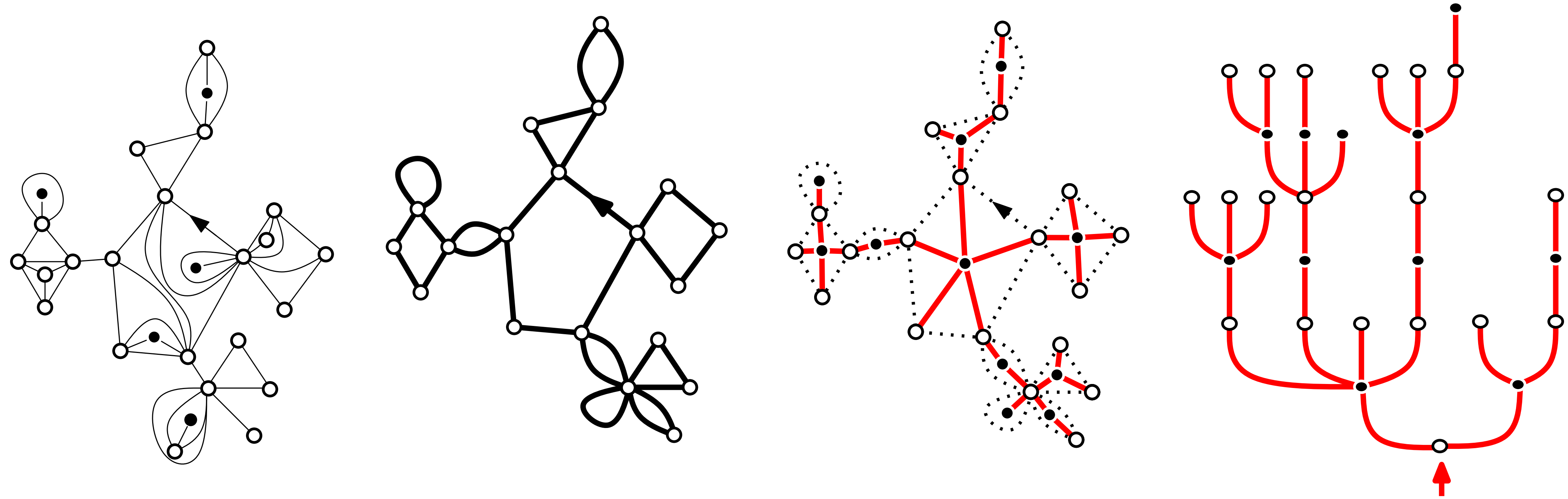
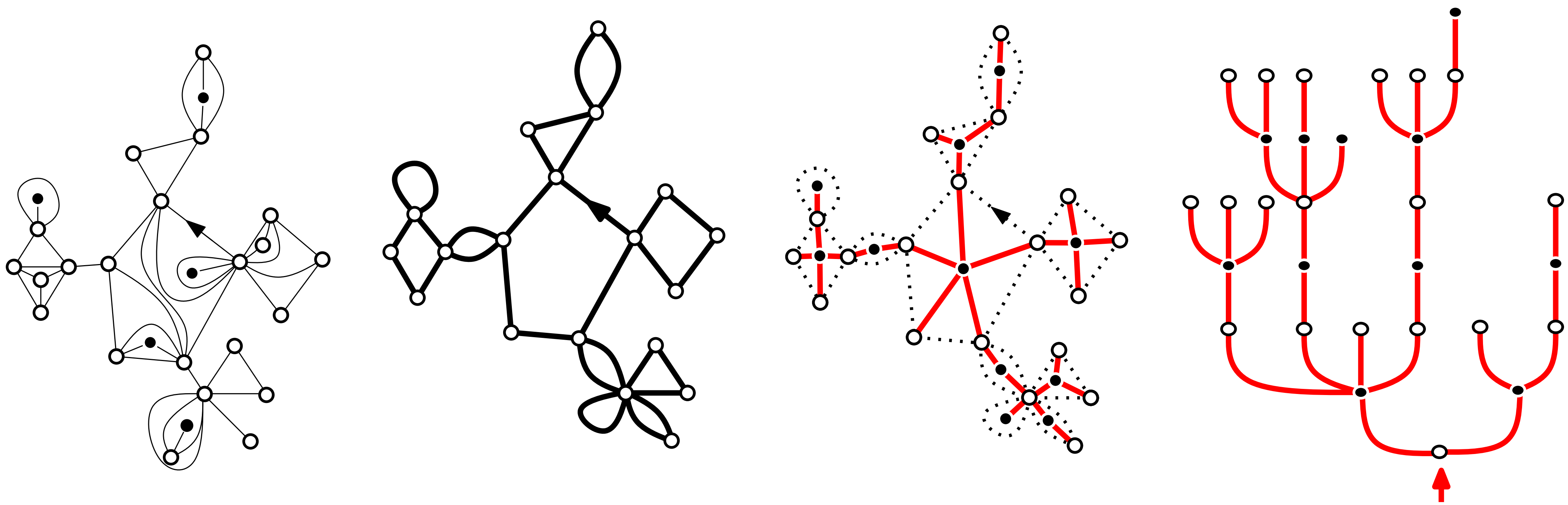


Figure: Construction of a two-type tree **Tree**$(\partial \mathcal{H})$.

Key Proposition (Curien & K.).

The tree **Tree**$(\partial \mathcal{H}_m^{(p)})$ is a two-type GW tree



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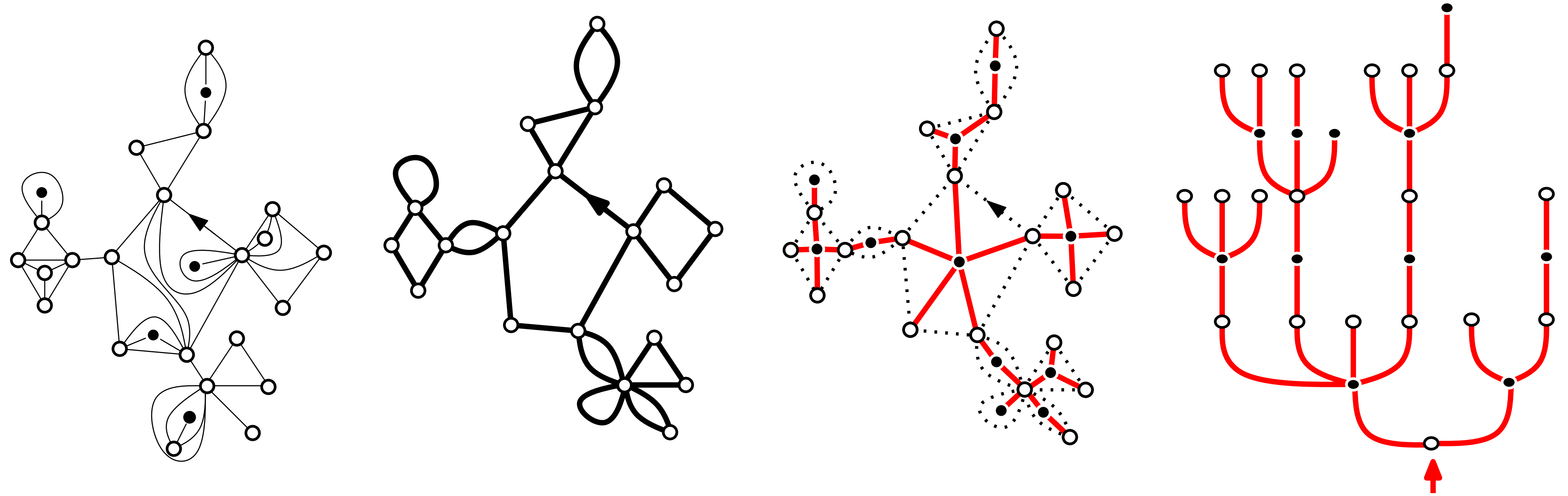
The tree **Tree**$(\partial\mathcal{H}_m^{(p)})$ is a two-type GW tree, conditioned on having $m+1$ vertices



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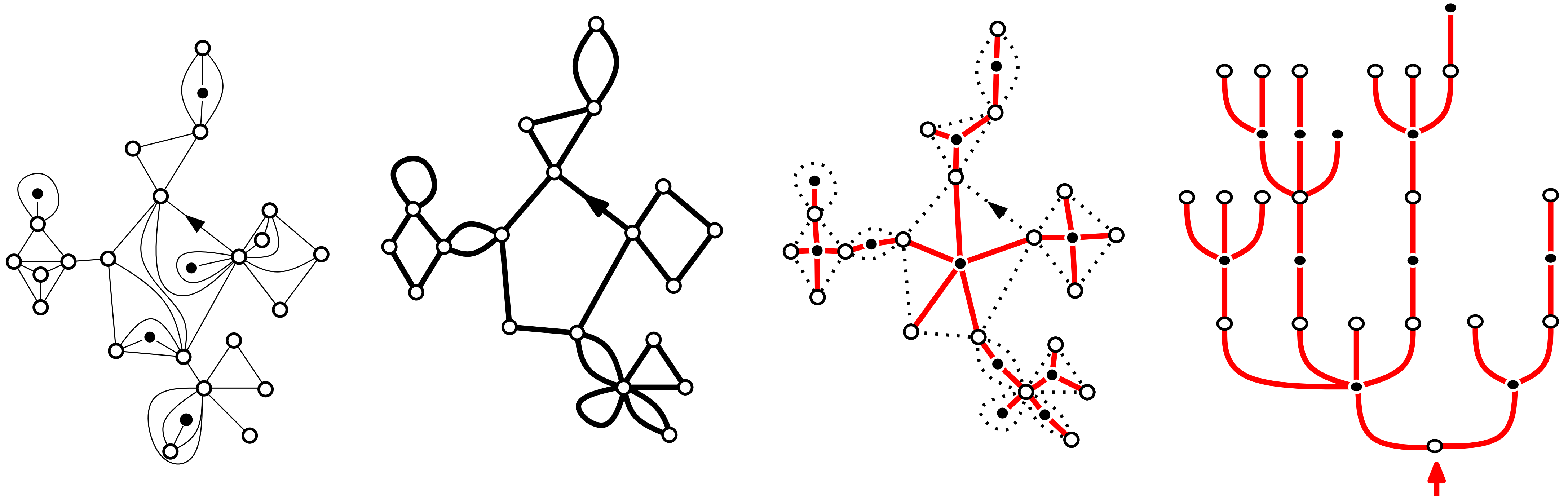
The tree **Tree**$(\partial \mathcal{H}_m^{(p)})$ is a two-type GW tree, conditioned on having $m+1$ vertices, whose offspring distributions are denoted by $\mu_\circ^{(p)}$ and $\mu_\bullet$.



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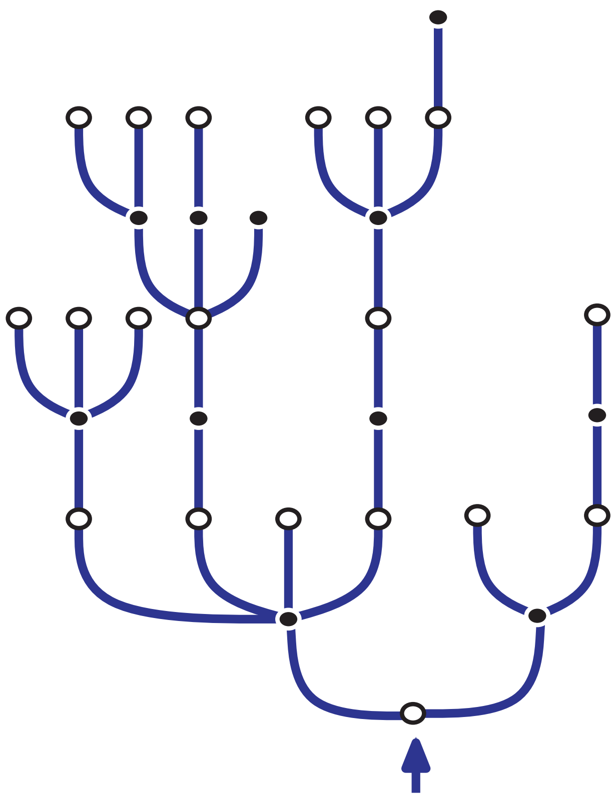
The tree **Tree**$(\partial\mathcal{H}_m^{(p)})$ is a two-type GW tree, conditioned on having $m+1$ vertices, whose offspring distributions are denoted by $\mu_\circ^{(p)}$ and $\mu_\bullet$. In addition $\mu_\circ^{(p)}$ is a geometric random variable.

Towards a one-type Galton–Watson tree



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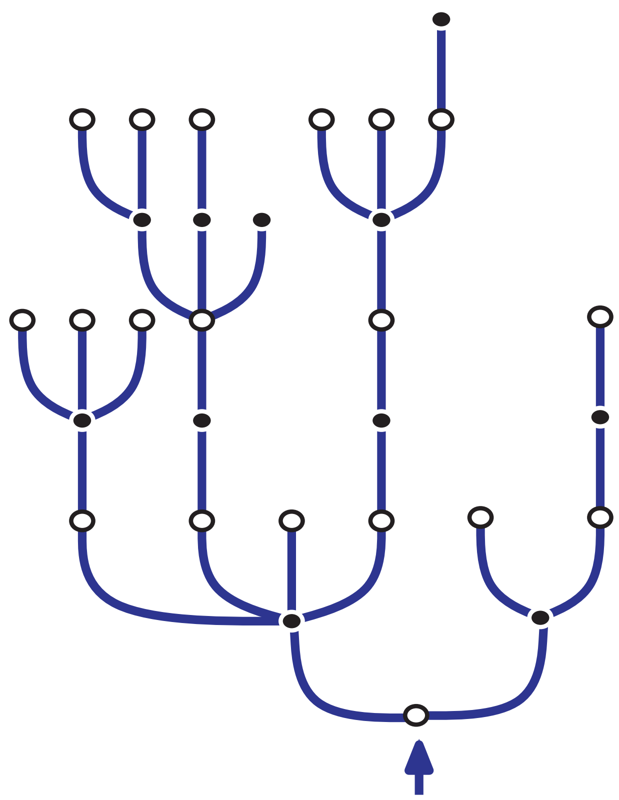

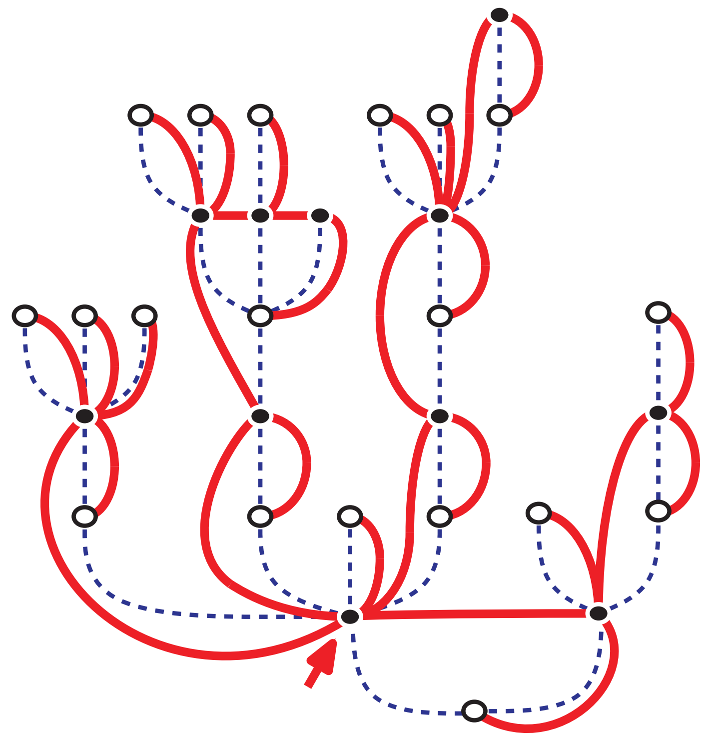

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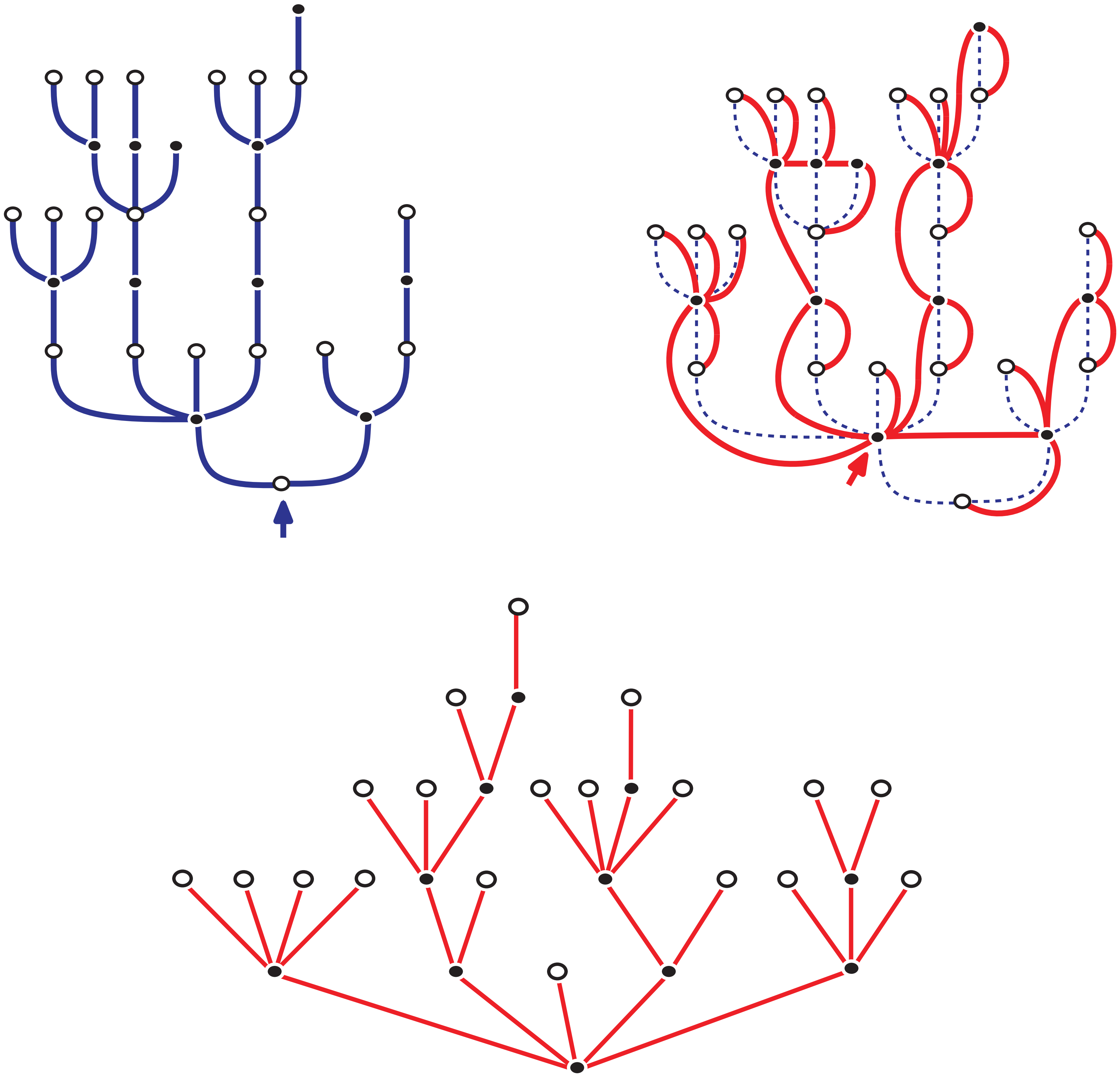

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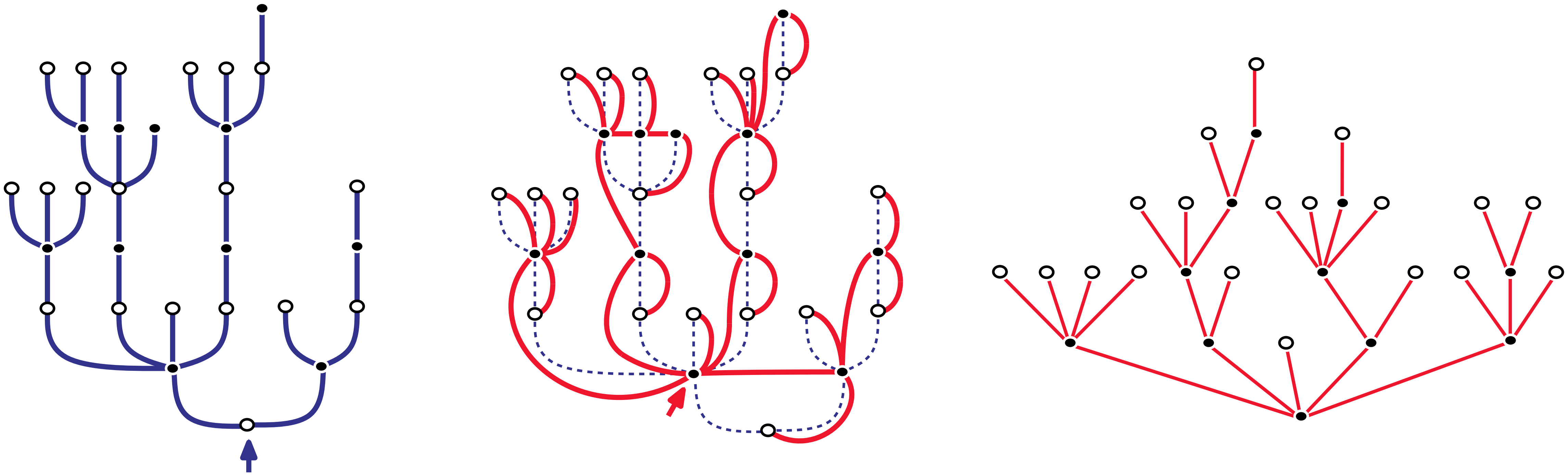

Figure: Construction due to Janson & Stefánsson of a tree $\mathcal{G}(\tau)$ from another tree τ.

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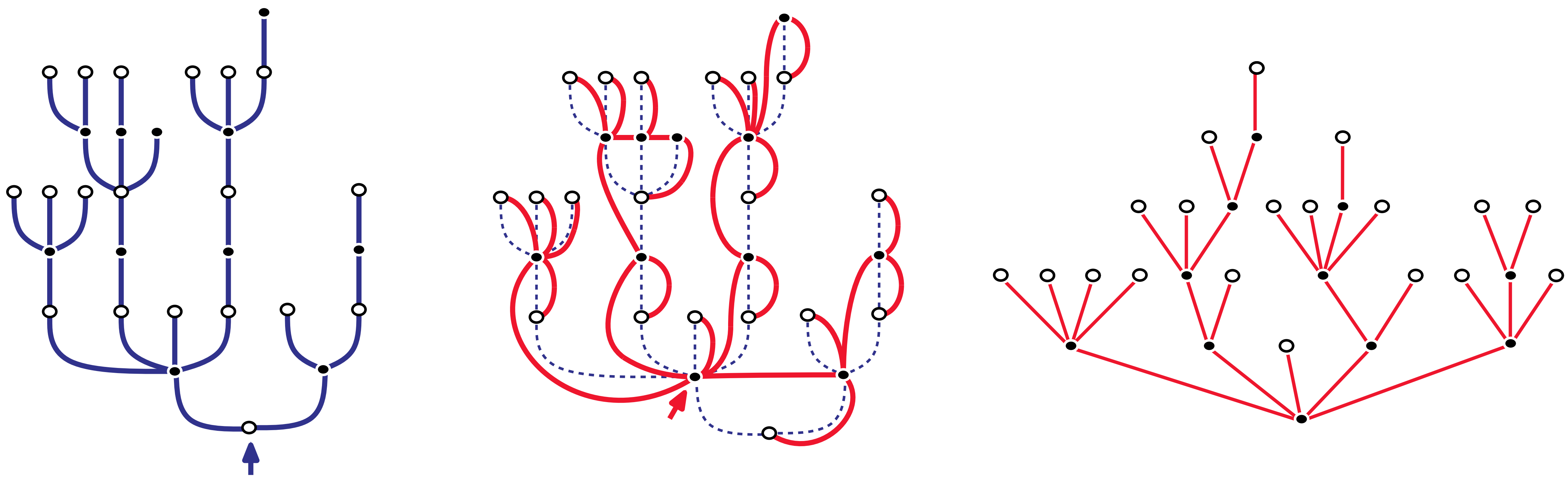

Figure: Construction due to Janson & Stefánsson of a tree $\mathcal{G}(\tau)$ from another tree τ.

Proposition (Janson & Stefánsson)

If $\mathfrak{t}$ is a two-type Galton–Watson tree such that $\mu_\circ$ is geometrical, then $\mathcal{G}(\mathfrak{t})$ is a one-type Galton–Watson tree.

Conclusion

Proposition (Curien & K.).

The tree $\mathcal{G}(\mathbf{Tree}(\partial\mathcal{H}_m^{(p)}))$ is a Galton–Watson tree

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- When $p = 1/2$, $\nu^{(1/2)}$ is **critical** and we have

$$\nu_i^{(1/2)} \quad \underset{i \to \infty}{\sim} \quad \frac{\sqrt{3}}{4\sqrt{\pi}} \cdot \frac{1}{i^{1+3/2}}.$$

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The theorems follow.