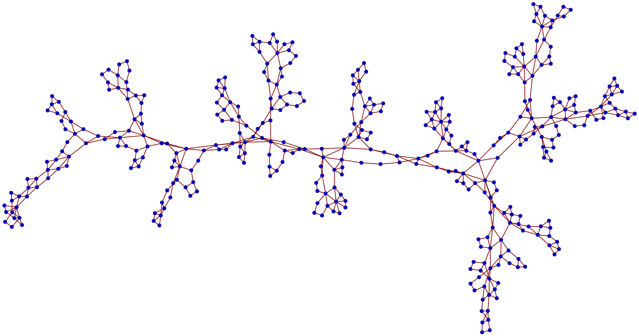
Scaling limits of random dissections



Igor Kortchemski (work with N. Curien and B. Haas)
DMA – École Normale Supérieure

SIAM Discrete Mathematics - June 2014

Outline

O. Motivation

I. Definition

II. Theorem

III. Application

IV. Proof

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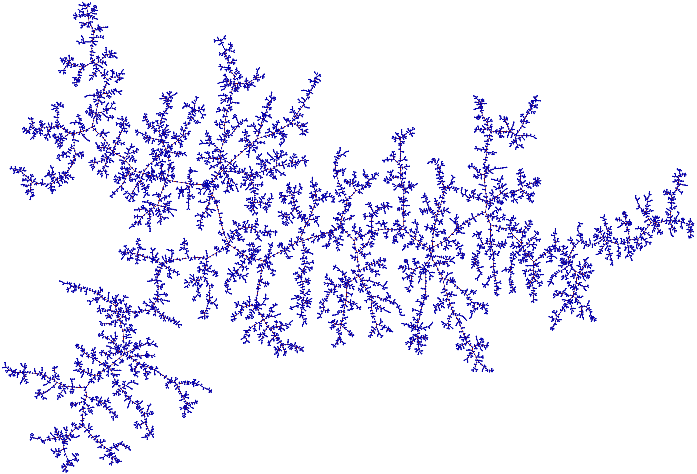
(see Le Gall's proceeding at ICM '14 for more information and references)

A simulation of the Brownian CRT



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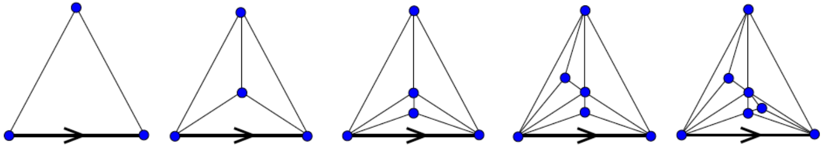

Figure : Figure by Albenque & Marckert

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O. Motivation

I. Definition: Boltzmann dissections

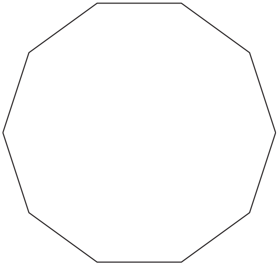
II. Theorem

III. Application

IV. Proof

Dissections

Let P_n be the polygon whose vertices are $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$.



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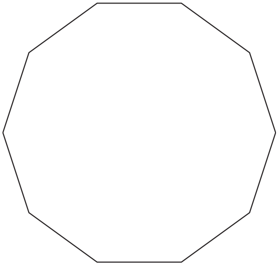

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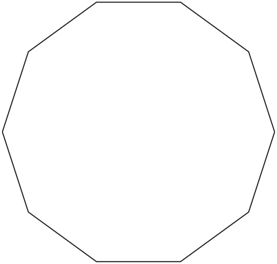

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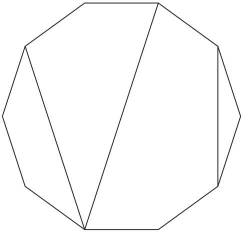
We will view dissections as **compact metric spaces**.

Dissections à la Boltzmann

Let $\mathbb{D}_n$ be the set of all dissections of P_n.



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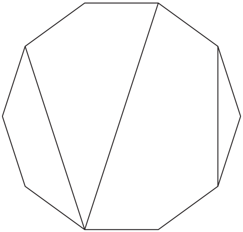

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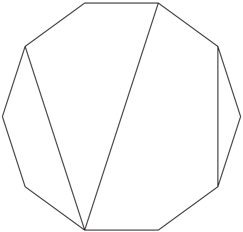

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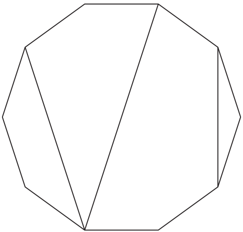

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Then define a probability measure on $\mathbb{D}_n$ by normalizing the weights:

$$Z_n = \sum_{\omega \in \mathbb{D}_n} \pi(\omega)$$

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We call $\mathbb{P}_n^{\mu}(\omega)$ a **Boltzmann probability distribution**.

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Then $\mathbb{P}_n^\nu = \mathbb{P}_n^\mu$ and ν is a critical probability measure on $\mathbb{Z}_+$ with $\nu_1 = 0$.

Dissections à la Boltzmann: examples

- **Uniform p-angulations ($p \geqslant 3$).** Set

$$\mu_0^{(p)} = 1 - \frac{1}{p-1}, \qquad \mu_{p-1}^{(p)} = \frac{1}{p-1}, \qquad \mu_i^{(p)} = 0 \quad (i \neq 0, i \neq p-1).$$

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III. Application

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What does $\mathcal{D}_n^\mu$ look like, for n large, **as a metric space**?

Simulations

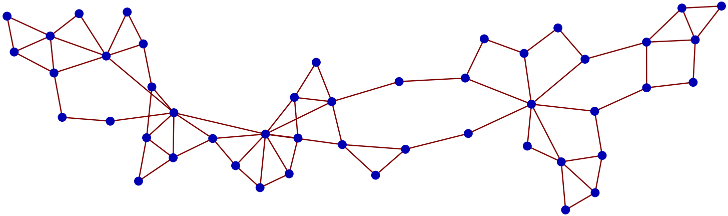

Figure : A uniform dissection of P_{45}.

Simulations

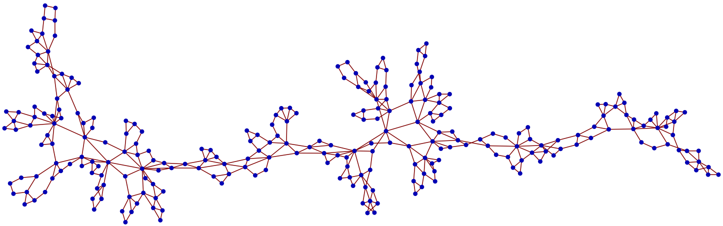

Figure : A uniform dissection of P_{260}.

Simulations

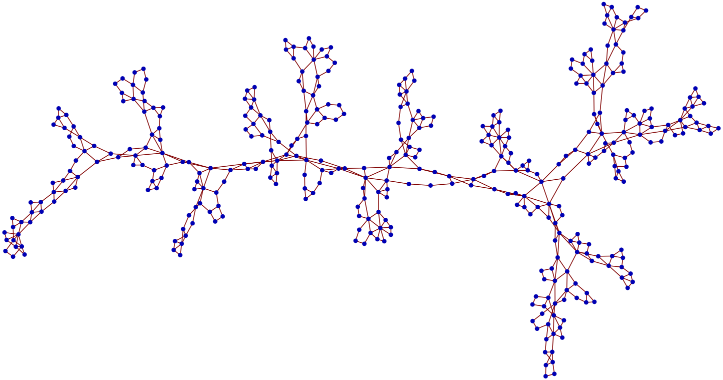

Figure : A uniform dissection of P_{387}.

Simulations

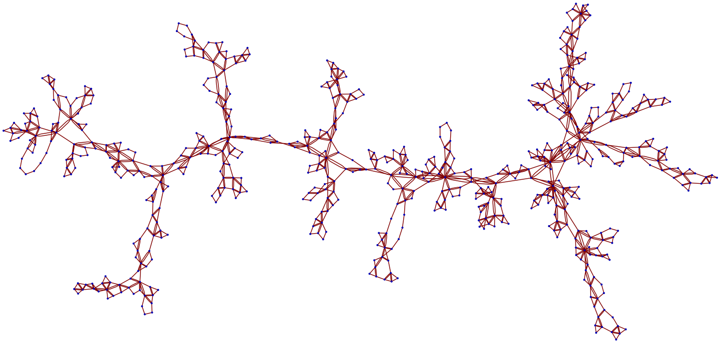

Figure : A uniform dissection of P_{637}.

Simulations

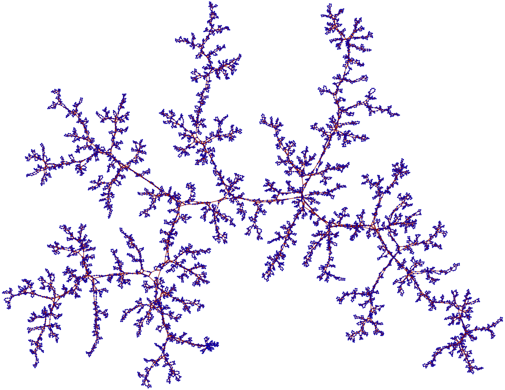

Figure : A uniform dissection of P_{8916}.

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Theorem (Curien, Haas & K.)**.**

There exists a constant $c(\mu)$ such that:

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where $\mu_{2\mathbb{Z}_+} = \mu_0 + \mu_2 + \mu_4 + \cdots$ and $\sigma^2 \in (0, \infty)$ is the variance of μ.

What is the Brownian Continuum Random Tree?

First define the contour function of a tree:

What is the Brownian Continuum Random Tree?

Knowing the contour function, it is easy to recover the tree by gluing:

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The Brownian tree $\mathcal{T}_{\mathbf{e}}$ is obtained by gluing from the Brownian excursion $\mathbf{e}$.

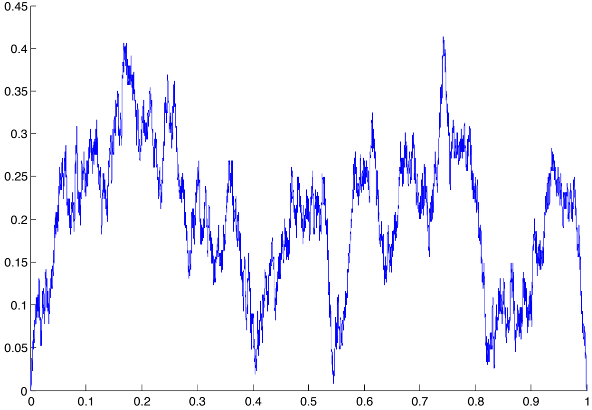

Figure : A simulation of $\mathbf{e}$.

A simulation of the Brownian CRT

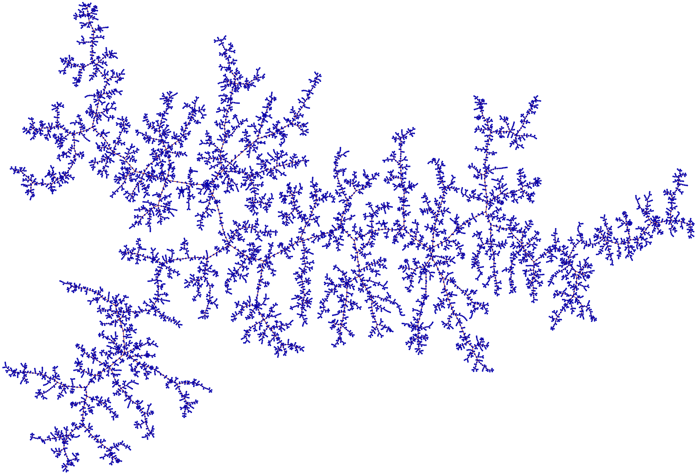

Figure : A non isometric plane embedding of a realization of $\mathcal{T}_{\mathbf{e}}$.

Recap

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I. Definition

II. Theorem

III. Combinatorial applications

IV. Proof

Applications

Combinatorial properties of random dissections have been studied by various authors :

- Uniform triangulations : Devroye, Flajolet, Hurtado, Noy & Steiger (maximal degree, longest diagonal, 1999) and Gao & Wormald (maximal degree, 2000),
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⇝ When $\mu_i \sim c/i^{1+\alpha}$ as $i \to \infty$, loops remain in the scaling limit, which is the stable looptree of index $\alpha \in (1,2)$ (Curien & K.). The looptree of index $\alpha = 3/2$ describes the scaling limit of boundaries of critical site percolation on the Uniform Infinite Planar Triangulation (Curien & K.).

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$\rightsquigarrow$ **Remark:** $c(\mu)$ is explicit for p-angulations and uniform dissections. For instance, for uniform dissections:

$$c(\mu) = \frac{1}{7}(3+\sqrt{2})2^{3/4}.$$

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For uniform dissections, we get $\mathbb{E}\left[\mathrm{Diam}(\mathcal{D}_n^\mu)\right] \sim \frac{1}{21}(3+\sqrt{2})2^{9/4} \sqrt{\pi n}$
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$$\underset{\simeq 0.74\sqrt{\pi n}}{\frac{(3+\sqrt{2})2^{1/4}}{7}\sqrt{\pi n}} \leqslant \quad \mathbb{E}\left[\mathrm{Diam}(\mathcal{D}_n^\mu)\right] \leqslant \quad \underset{\simeq 1.5\sqrt{\pi n}.}{2 \cdot \frac{(3+\sqrt{2})2^{1/4}}{7}\sqrt{\pi n}}$$

I. Definition

II. Theorem

III. Application

IV. Proof

Proof

Step 1. Consider the dual tree of $\mathcal{D}_n^\mu$:

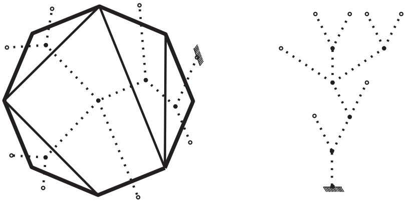

Figure : A dissection and its dual tree $\mathcal{T}_n^\mu$.

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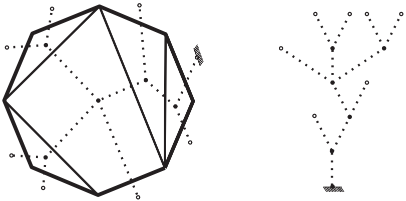

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Key fact. $\mathcal{T}_n^{\mu}$ is a (planted) Galton–Watson tree with offspring distribution μ conditioned on having $n-1$ leaves.

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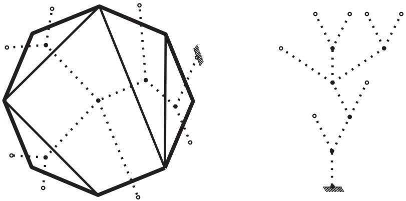

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Key fact. $\mathcal{T}_n^{\mu}$ is a (planted) Galton–Watson tree with offspring distribution μ conditioned on having $n-1$ leaves. It is known (Rizzolo or K.) that:

$$\frac{1}{\sqrt{n}} \cdot \mathcal{T}_n^{\mu} \quad \xrightarrow[n\to\infty]{(d)} \quad \frac{2}{\sigma\sqrt{\mu_0}} \cdot \mathcal{T}_{\mathbf{e}}.$$

Proof

➳ **Step 2**. We show that:

$$\mathcal{D}_n^{\mu} \quad \simeq \quad \frac{1}{4}\left(\sigma^2 + \frac{\mu_0\mu_{2\mathbb{Z}_+}}{2\mu_{2\mathbb{Z}_+} - \mu_0}\right) \cdot \mathcal{T}_n^{\mu}.$$

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To this end, we compare the length of geodesics in $\mathcal{D}_n^\mu$ and in $\mathcal{T}_n^\mu$ by using an "exploration" Markov Chain:

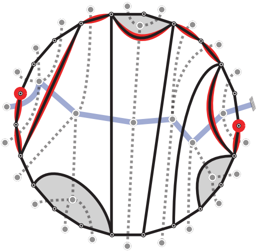
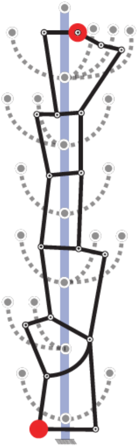


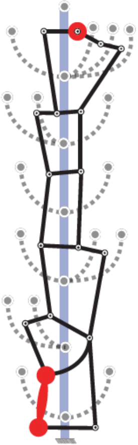
Figure : A geodesic in $\mathcal{T}_n^\mu$ (in light blue) and the associated geodesic in $\mathcal{D}_n^\mu$ (in red).

The Markov Chain



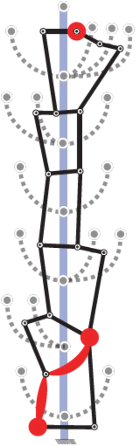
$i = 0$

The Markov Chain



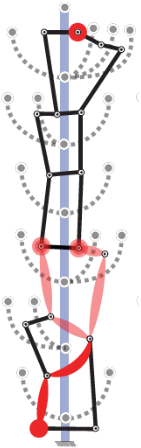
$i = 1$

The Markov Chain



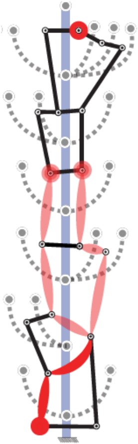
$i = 2$

The Markov Chain



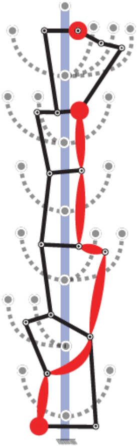
$i = 3$

The Markov Chain



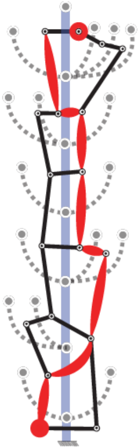
$i = 4$

The Markov Chain



$i = 5$

The Markov Chain



$i = 6$