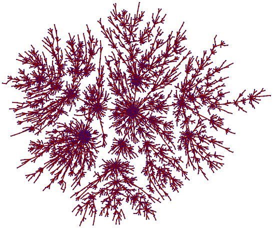
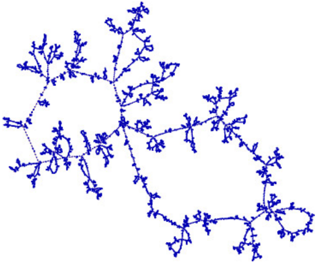
The Brownian Rabbit

and

scaling limits of preferential attachment trees





Igor Kortchemski (joint work with N. Curien, T. Duquesne & I. Manolescu)
DMA – École Normale Supérieure – Paris

44ème École d'été de probabilités de Saint-Flour, July 2014

Outline

I. Preferential attachment and influence of the seed

II. Looptrees and preferential attachment

III. Extensions and conjectures

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This is the preferential attachement model (Szymánski '87; Albert & Barabási '99; Bollobás, Riordan, Spencer & Tusnády '01).

Influence of the seed

Question (Bubeck, Mossel & Rácz): What is the influence of the seed tree?

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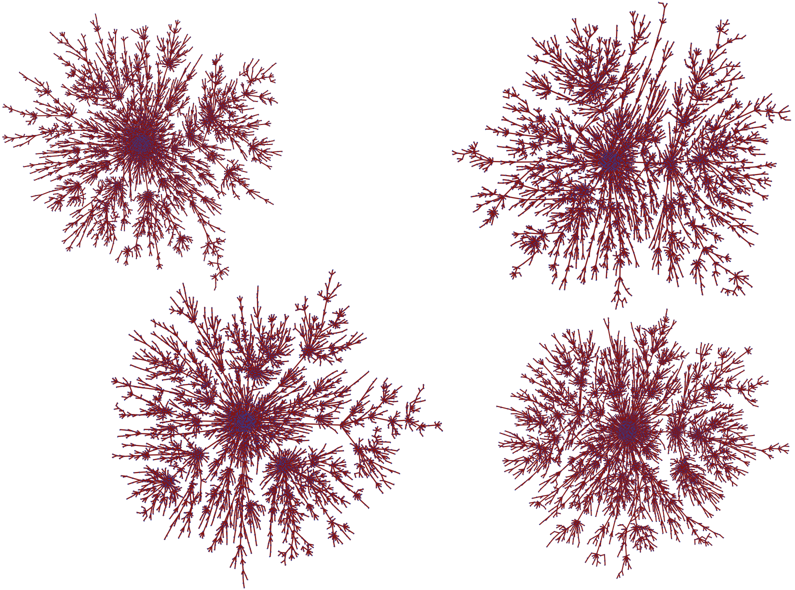
Four simulations of $T_n^{(S_1)}$ for $n = 5000$:



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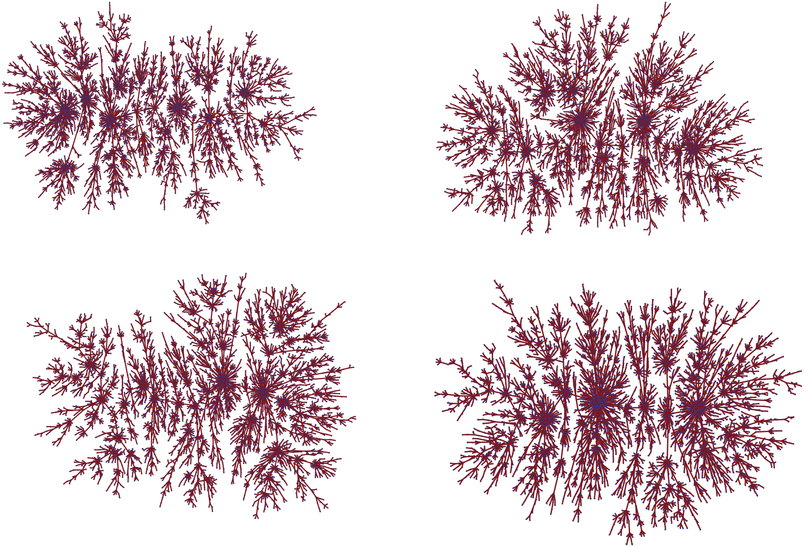

Referendum

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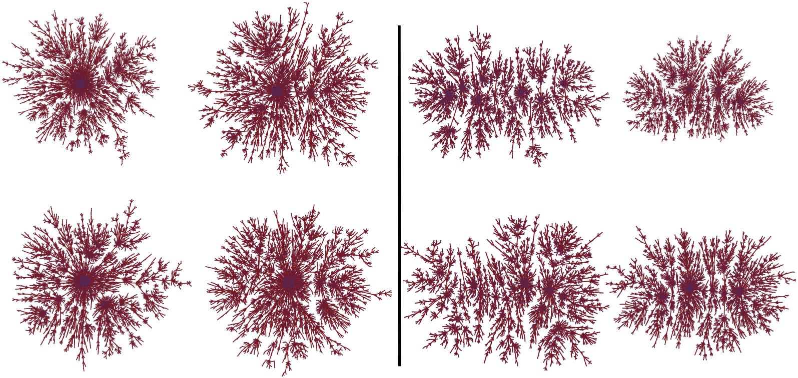

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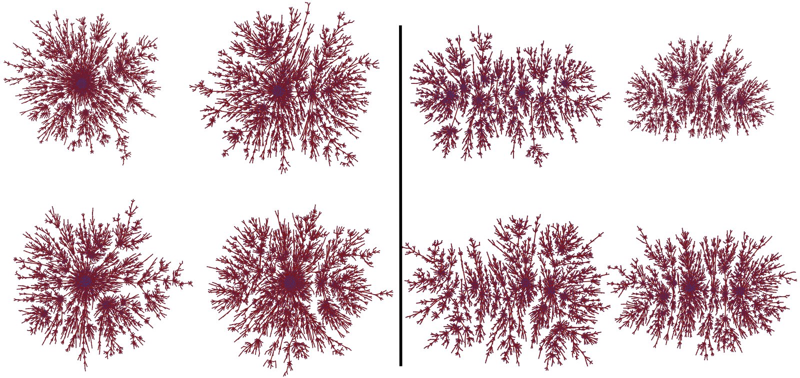

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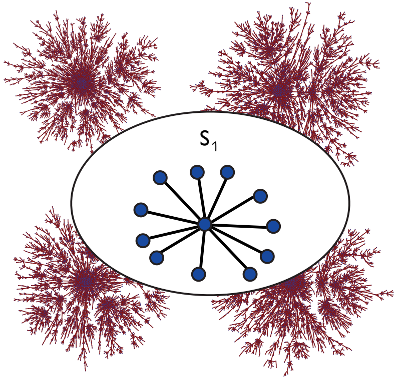

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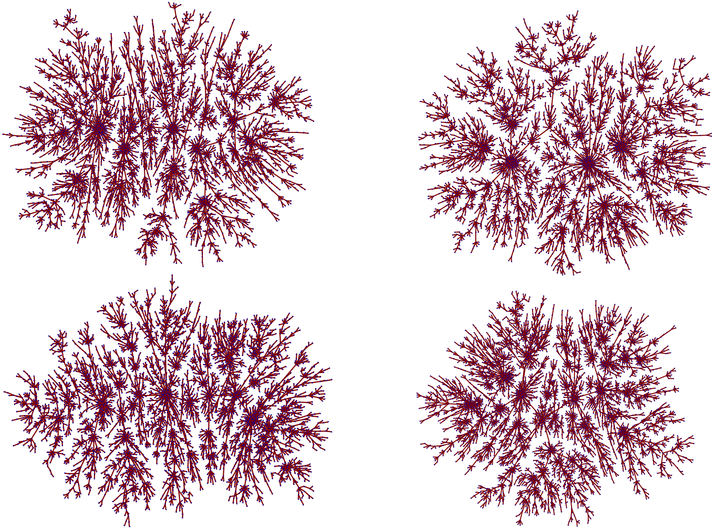

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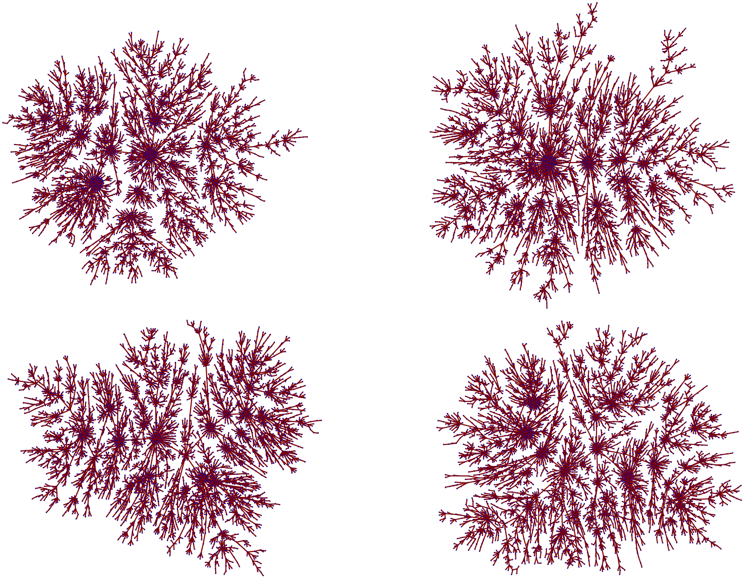

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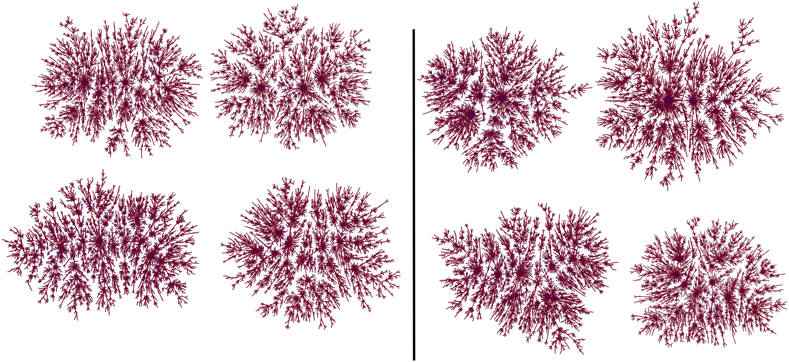

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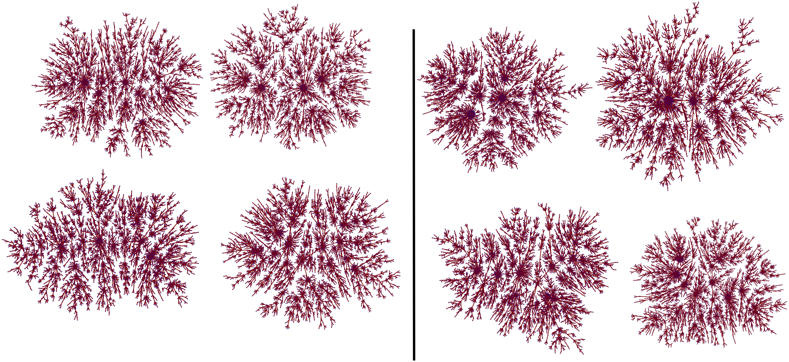

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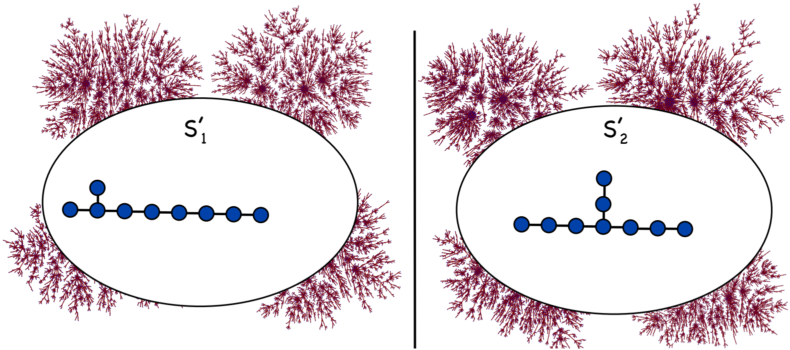

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Influence of the seed graph

For finite trees S_1 and S_2, set

$$d(S_1, S_2) = \lim_{n\to\infty} d_{TV}(T_n^{(S_1)}, T_n^{(S_2)}),$$

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Theorem (Curien, Duquesne, K. & Manolescu '14)**.**

The conjecture is true.

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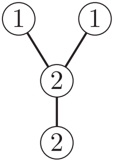
$\leadsto$ The graph structure of $T_n^{(S)}$ is that of preferential attachment.

Our observables: embeddings of decorated trees

A decorated tree τ is a tree τ with positive integer labels $(\ell(u), u \in \tau)$.



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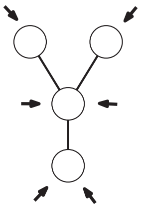

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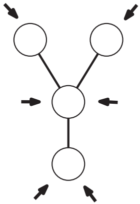

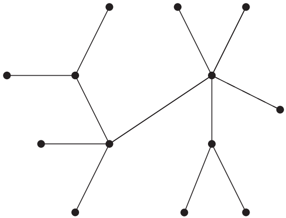

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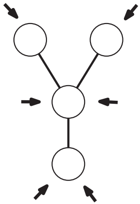
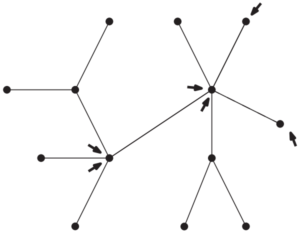
I.e. $D_\tau(T)$ is the number of ways to embed τ in T s.t. each arrow pointing to a vertex of τ is associated with a corner of T adjacent to the corresponding vertex (distinct arrows associated with distinct corners).





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Hence there exist constants α_n, β_n such that

$$M_2(n) = \alpha_n D_{\tau_2}(T_n^{(S)}) - \beta_n$$

is a martingale.

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There exists a partial order $\preccurlyeq$ on decorated trees, such that for every decorated tree $\boldsymbol{\tau}$, there exist constants $\{c_n(\boldsymbol{\tau}, \boldsymbol{\tau'}) : \boldsymbol{\tau'} \preccurlyeq \boldsymbol{\tau}, n \geqslant 2\}$ such that, for every seed S,

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I. Preferential attachment and influence of the seed

II. Looptrees and preferential attachment

III. Extensions and conjectures

⇝ Question. Does the sequence $(T_n^{(S)})$ admit scaling limits?

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⇝ Answer: no.

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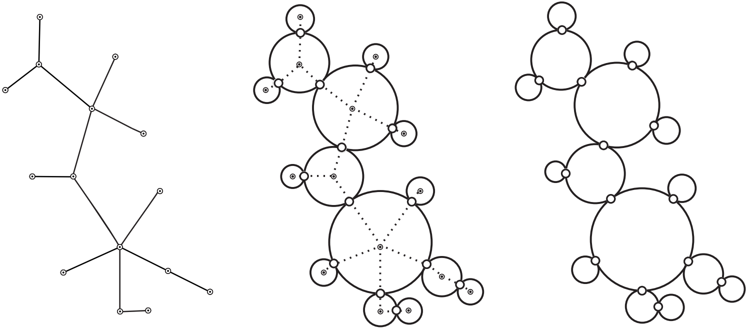

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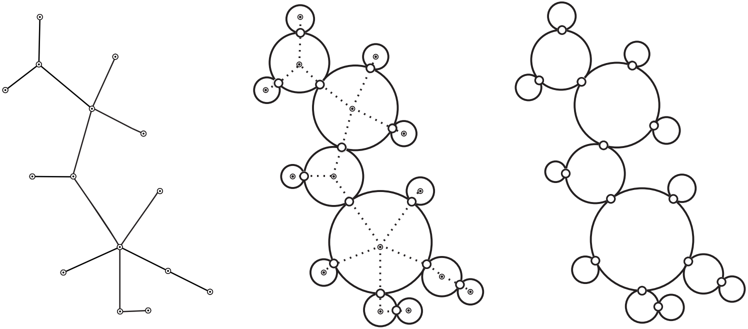

Figure : A plane tree τ and its associated discrete looptree $\mathrm{Loop}(\tau)$.

We view $\mathrm{Loop}(\tau)$ as a compact metric space.

Scaling limits of trees built by preferential attachment

Theorem (Curien, Duquesne, K., Manolescu)**.**

There exists a random compact metric space $\mathcal{L}^{(S)}$ such that:

$$n^{-1/2} \cdot \mathrm{Loop}(T_n^{(S)}) \quad \xrightarrow[n \to \infty]{a.s.} \quad \mathcal{L}^{(S)},$$

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We will see that $n^{1/2}$ is the order of large degrees in $T_n^{(S)}$.

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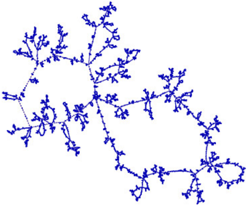

Figure : The looptree of a large tree built by preferential attachement.

Main idea: coupling with Rémy's algorithm

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Proposition (Rémy '85)

For every fixed $n \geqslant 1$, the tree $\mathbf{B}_n$ is uniformly distributed over the set of all binary trees with $n+1$ labeled leaves.

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Useful notation: for $1 \leqslant i \leqslant n$, denote by $\mathrm{Span}(B_n; A_0, A_1, \ldots, A_{i-1})$ the subtree of B_n spanned by the leaves $A_0, A_1, \ldots, A_{i-1}$.

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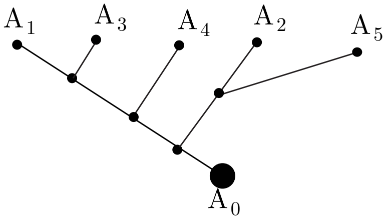

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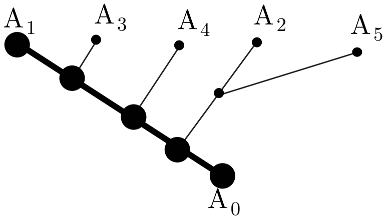

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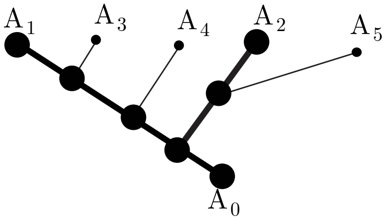

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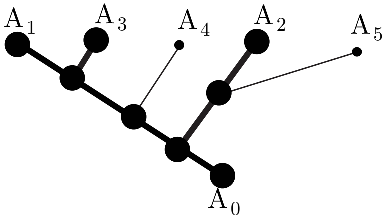

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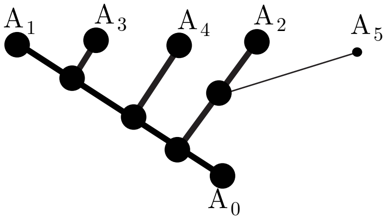

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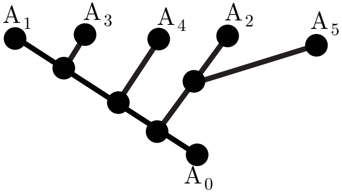

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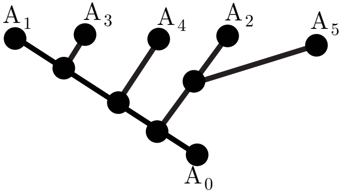

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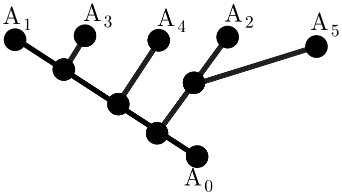

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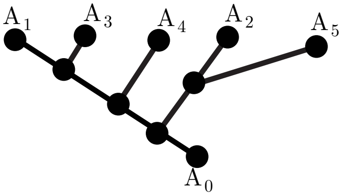

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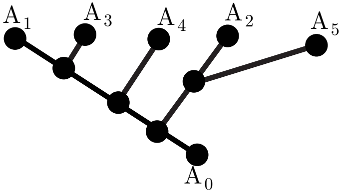

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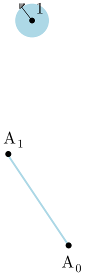

Step: n=1

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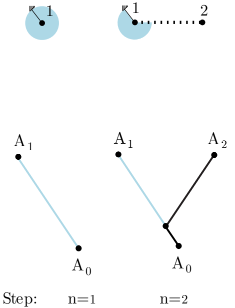

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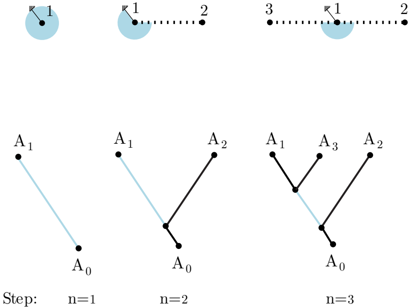

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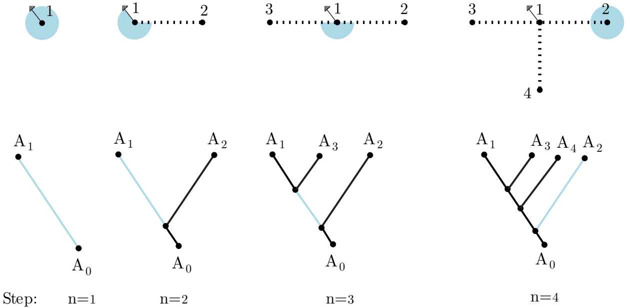

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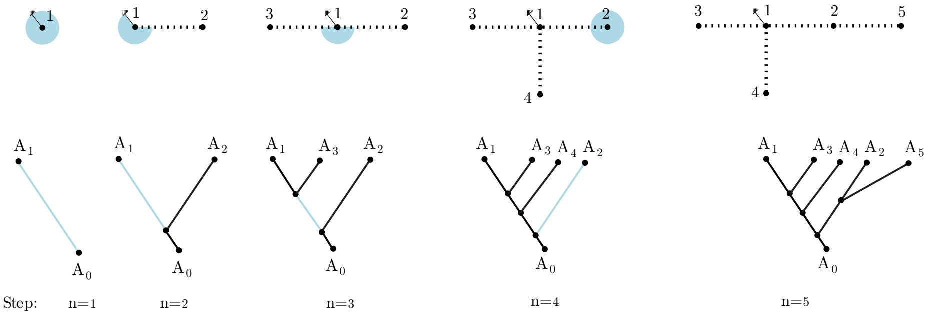

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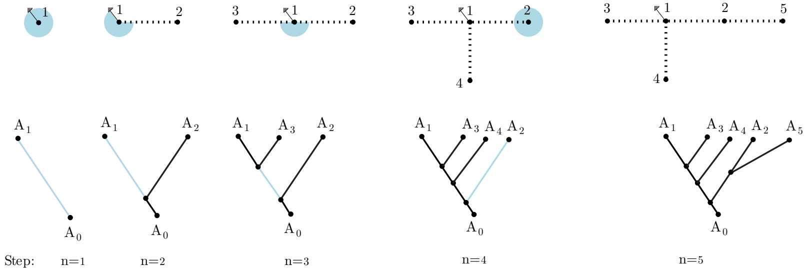

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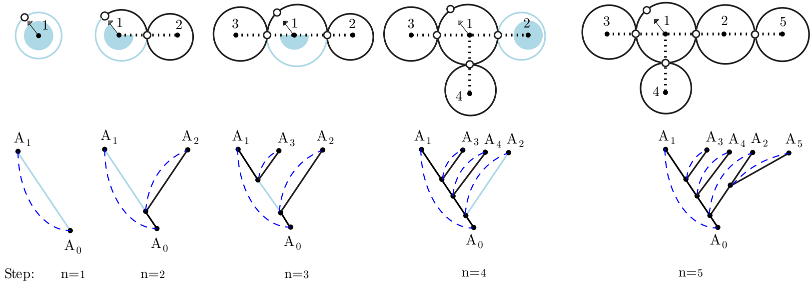
We have:

$$(\mathrm{Loop}(\mathsf{T}_n^{-\circ}); n \geqslant 1) \overset{(d)}{=} (\mathrm{Glu}(\mathbf{B}_n); n \geqslant 1).$$



Recall that $\mathbf{B}_n = (\mathrm{B}_n; A_0, A_1, \ldots, A_n)$. Let $\mathrm{Glu}(\mathbf{B}_n)$ be the graph obtained from B_n by identifying:

- A_1 with A_0
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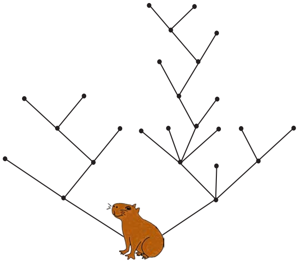
We have:

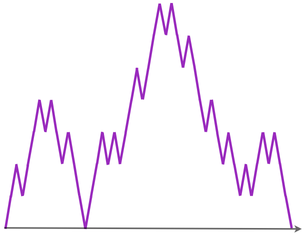
$$(\mathrm{Loop}(T_n^{\multimap}); n \geqslant 1) \quad \overset{(d)}{=} \quad (\mathrm{Glu}(\mathbf{B}_n); n \geqslant 1).$$

$\leadsto$ **Key fact:** Rémy's algorithm converges to the Brownian Continuum Random Tree.

What is the Brownian Continuum Random Tree?

First define the contour function of a tree:





What is the Brownian Continuum Random Tree?

Knowing the contour function, it is easy to recover the tree by gluing:

(animation here)

What is the Brownian Continuum Random Tree?

The Brownian tree $\mathcal{T}$ is obtained by gluing from the Brownian excursion $\mathbf{e}$.

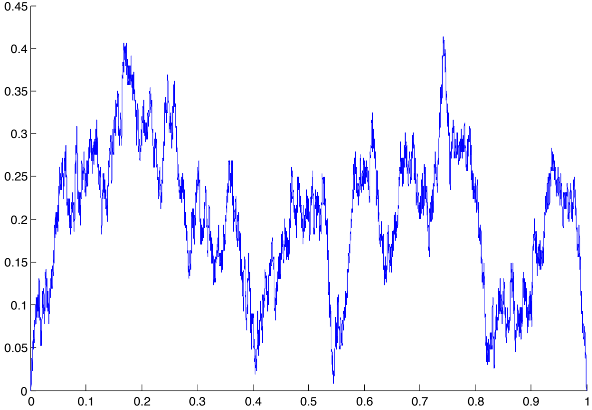


Figure : A simulation of $\mathbf{e}$.

A simulation of the Brownian CRT

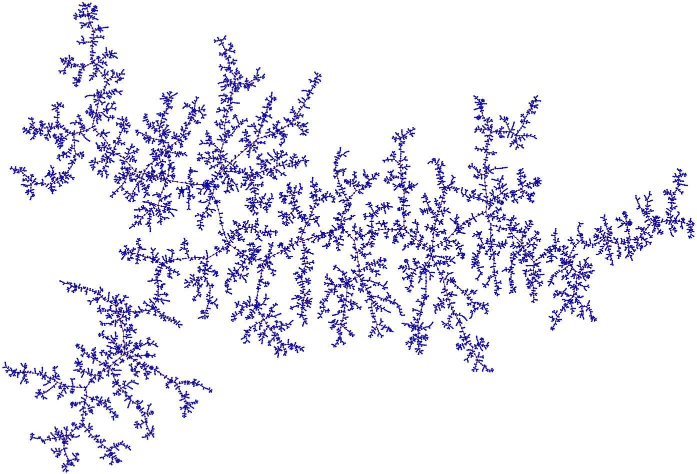


Figure : A non isometric plane embedding of a realization of $\mathcal{T}_{\mathbf{e}}$.

Proposition (Curien & Haas '13)

There exists a pair $(\mathcal{T}_{\mathbf{e}}, (X_i; i \geqslant 0))$, where $\mathcal{T}_{\mathbf{e}}$ is a Brownian CRT and $(X_i; i \geqslant 0)$ is a collection of i.i.d. vertices sampled according to its mass measure

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and hence

$$n^{-1/2} \cdot \mathrm{Loop}(\mathrm{T}_n^{\multimap}) \quad \xrightarrow[n\to\infty]{a.s.} \quad \mathcal{L},$$

I. Preferential attachment and influence of the seed

II. Looptrees and preferential attachment

III. Extensions and conjectures

What happens for Galton–Watson trees?

Let μ be a critical ($\sum_{i \geqslant 0} i\mu_i = 1$) probability measure on $\{0, 1, 2, \ldots\}$ and let $\mathcal{T}_n$ be a μ-Galton–Watson tree conditioned to have n vertices.

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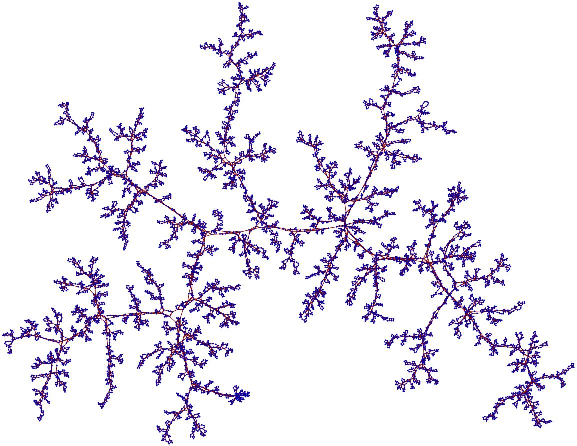

Figure : A non isometric plane embedding of a realization of a looptree of a large critical Galton–Watson tree with finite variance.

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and is called the stable looptree of index α. In addition, $\mathcal{L}_{3/2}$ is the scaling limit of the boundary of (critical) site percolation on Angel & Schramm's Uniform Infinite Planar Triangulation.

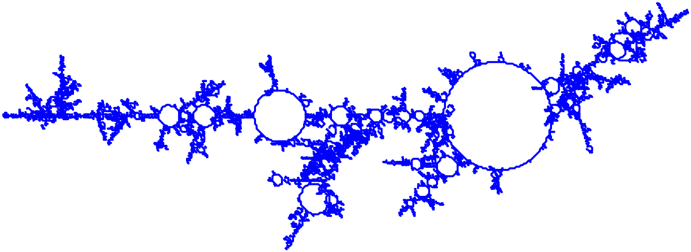

Figure : A non isometric plane embedding of a realization of $\mathcal{L}_{3/2}$, the stable looptree of index 3/2.

Conjectures: back to preferential attachment

Question.

What happens for linear preferential attachment, i.e. when instead of being chosen proportionally to $\deg(u)$ a vertex u is chosen proportionally to $\deg(u) + a$ with $a > -1$?

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Conjecture.

For every plane trees S_1, S_2, we have

$$\lim_{n\to\infty} d_{TV}(T_n^{(S_1)}, T_n^{(S_2)}) = d_{TV}(\mathcal{L}^{(S_1)}, \mathcal{L}^{(S_2)}).$$