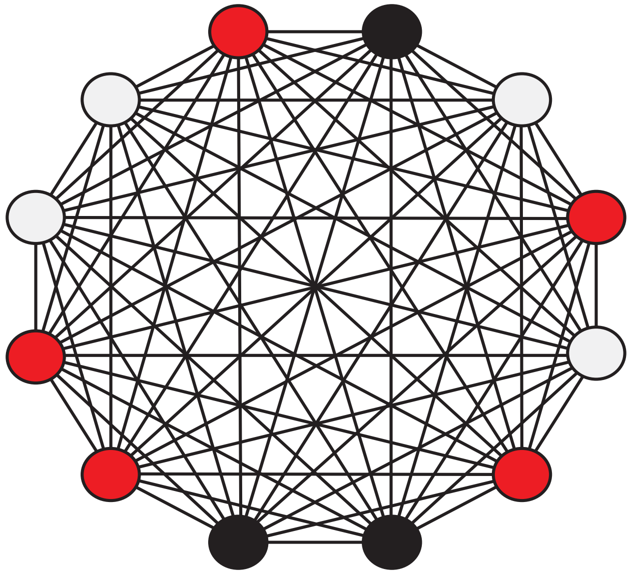
On a prey-predator model



Igor Kortchemski
CNRS & CMAP, École polytechnique

Bonn probability seminar – July 2015

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vacant vertex $\Longleftrightarrow$

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vacant vertex	$\Longleftrightarrow$	vacant vertex
prey	$\Longleftrightarrow$	healthy cell
predator	$\Longleftrightarrow$	cell infected by a virus

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vacant vertex	$\Longleftrightarrow$	normal individual
prey	$\Longleftrightarrow$	individual trying to spread a rumor (spreader)
predator	$\Longleftrightarrow$	individual trying to scotch the rumor (stifler)

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vacant vertex	$\Longleftrightarrow$	Susceptible (S) individual
prey	$\Longleftrightarrow$	Infected (I) individual
predator	$\Longleftrightarrow$	Recovered (R) individual

Here, $\{I, S\} \xrightarrow{\lambda} \{I, I\}, \quad \{R, I\} \xrightarrow{1} \{R, R\}$.

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- Maki–Thompson ('73) directed rumour propagation model, where $(\mathrm{I},\mathrm{S}) \xrightarrow{1} (\mathrm{I},\mathrm{I}), \quad (\mathrm{R},\mathrm{I}) \xrightarrow{1} (\mathrm{R},\mathrm{R}), \quad (\mathrm{I},\mathrm{I}) \xrightarrow{1} (\mathrm{I},\mathrm{R})$.

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- Williams Bjerknes ('71) tumor growth model (or biased voter model), where $(I, S) \xrightarrow{\lambda} (I, I), \quad (S, I) \xrightarrow{1} (S, S)$.

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- Kordzakhia ('05), where $\{I, S\} \xrightarrow{\lambda} \{I, I\}, \quad \{R, I\} \xrightarrow{1} \{R, R\}, \quad \{R, S\} \xrightarrow{1} \{R, R\}$.

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FACEBOOK

Facebook Is About to Lose 80% of Its Users, Study Says

Social media is like a disease that spreads, and then dies

By Sam Frizell @Sam_Frizell | Jan. 21, 2014 | 266 Comments

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Facebook's growth will eventually come to a quick end, much like an infectious disease that spreads rapidly and suddenly dies, say Princeton researchers who are using diseases to model the life cycles of social media.

Facebook To **Lose** 80% Of Users By 2017
InformationWeek - 23 janv. 2014
Online social networks spread like disease epidemics, and **Facebook** will **lose** 80% of its victims -- I mean, users -- by 2017, according to a study from Princeton University researchers. The study, "Epidemiological modeling of online social network dynamics" ...

Facebook could **lose** 80 percent of users by 2017, report claims
Fox News - 23 janv. 2014
"**Facebook** has already reached the peak of its popularity and has entered a decline phase," they concluded. "The future suggests that **Facebook** will undergo a rapid decline in the coming years, **losing** 80 percent of its peak user base between 2015 and 2017 ...

Facebook will **lose** 80 percent of its users in next 4 years, Princeton study says
The Star-Ledger - NJ.com - 23 janv. 2014
Most of the 874 million people across the world who sign on to **Facebook** will stop doing so in the next four years, according to a Princeton University study. The study predicts the social media site will **lose** 80 percent of the users it had at its 2010 peak ...

Facebook Losing Users; 30 Years of Mac Ads; Snapchat 'Ghost' Verification
PC Magazine - 23 janv. 2014
Topping tech headlines Wednesday, a new study predicts a rapid decline for **Facebook**, which researchers said will **lose** 80 percent of its peak user base between 2015 and 2017. Using epidemiological models to track the spread of infectious diseases and ...

Facebook Might **Lose** 80% of Users and be the Next 'MySpace,' Study Says
Morning Ledger - 23 janv. 2014
Facebook Might **Lose** 80 percent and be the Next MySpace A new study conducted and released by Princeton University has described social networks as similar to infectious diseases. It pointed out that such sites gain millions of users within just a short span ...

Facebook Will **Lose** 80 Percent of Users by 2017
Guardian Liberty Voice - 23 janv. 2014
Facebook According to researchers at Princeton University, **Facebook** will **lose** 80 percent of its users by 2017. The researchers have also stated that that decline is already happening now and could reach the total any time within 2015 and the 2017 deadline.

Facebook to '**lose** 80% of users by 2017'
Irish Times - 23 janv. 2014
Facebook has spread like an infectious disease but we are slowly becoming immune to its attractions, and the platform will be largely abandoned by 2017, say researchers at Princeton University. The forecast of **Facebook's** impending doom was made by ...

Facebook will **LOSE** 80% of its users by 2017 – epidemiological study
Register - 23 janv. 2014
According to the students' paper, **Facebook** is "just beginning to show the onset of an abandonment phase", after reaching its popularity peak in 2012, which will lead to it **losing** 80 per cent of its peak user base between 2015 and 2017. The paper, which has ...

Facebook Predicts Princeton Won't Exist In 2021
InformationWeek - 24 janv. 2014
Princeton's report, from the university's Department of Mechanical and Aerospace Engineering, used Google search data to predict engagement trends, ultimately concluding that **Facebook** was set to **lose** a whopping 80% of users by 2017. Such a ...

Could **Facebook** Really **Lose** 80% of its Users?
DailyFinance - 23 janv. 2014
Facebook has so far been the only super-hot social media network to escape the fate of former top sites like MySpace, Friendster, or even GeoCities/Tripod back in the day. And with its now-successful stock offering and seeming ubiquity among nearly every ...

Facebook Will **Lose** 80 Percent Of Users In Next Three Years, Researchers Say
Opposing Views - 23 janv. 2014
People are slowly building up an immunity to **Facebook** and researchers predict it will **lose** 80 percent of its peak user base by 2017. Researchers at Princeton University compared the growth of the social media site to the spread of disease. They believe ...

Outline

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Figure: Excerpt of the film "Gunfight at the O.K. Corral" (1957)

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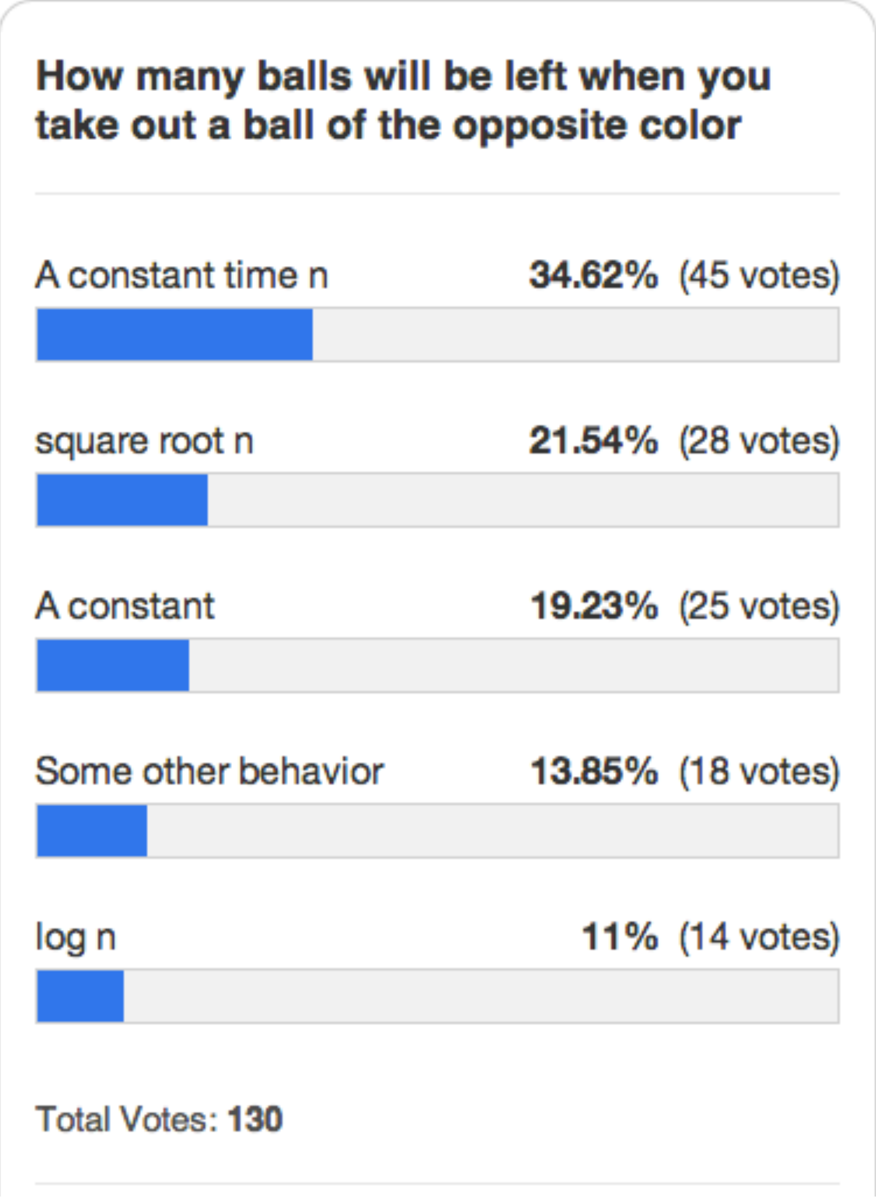

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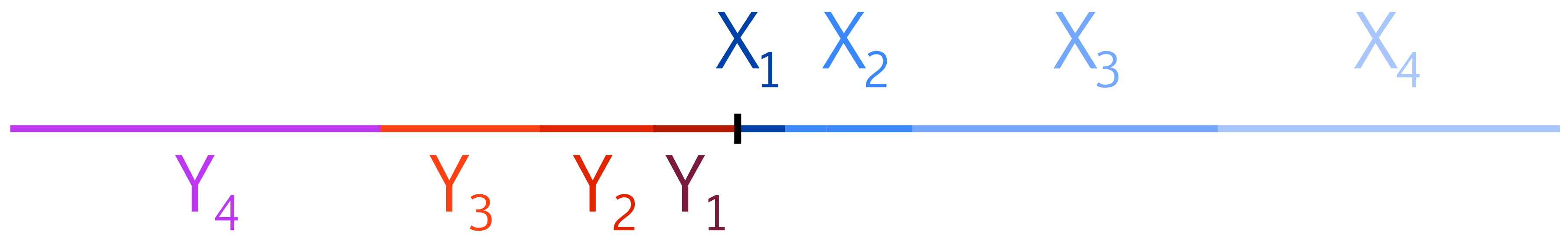
$$\text{length}([i-1, i]) = X_i, \quad \text{length}([-i, -i+1]) = Y_i \qquad (1 \leqslant i \leqslant n).$$



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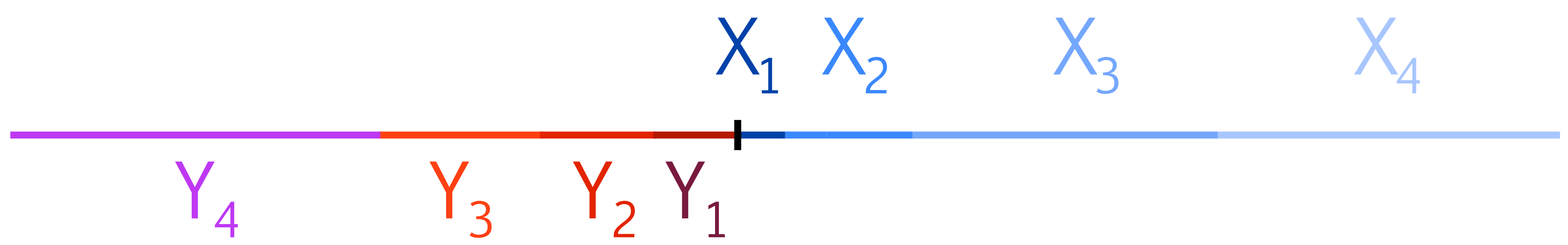

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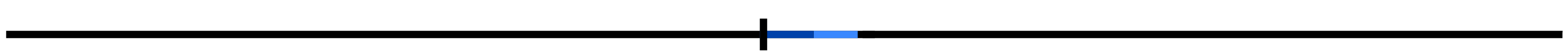

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Light both ends, and stop the fire when the origin is reached. Let $R(n)$ be the number of remaining pieces. Then $R(n)$ has the same law as the number of remaining balls in the urn/gunman problem.

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In order to estimate the number $R(n)$ of remaining pieces, first estimate the remaining length $L(n)$:

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Set $S_k = X_1 + \cdots + X_k$. We have $\mathbb{E}[S_k] \simeq k^2$, so $S_k \simeq k^2$. But, if the left part burns first, $S_{R(n)} \simeq L(n)$. Hence

$$R(n)^2 \simeq n^{3/2}$$

so that $R(n) \simeq n^{3/4}$.

This "decoupling" idea is called the Athreya–Karlin embedding, and is useful to study more general Pólya urn schemes.

I. Test your intuition!

II. Prey & predators on a complete graph

III. Preys & predators on an infinite tree

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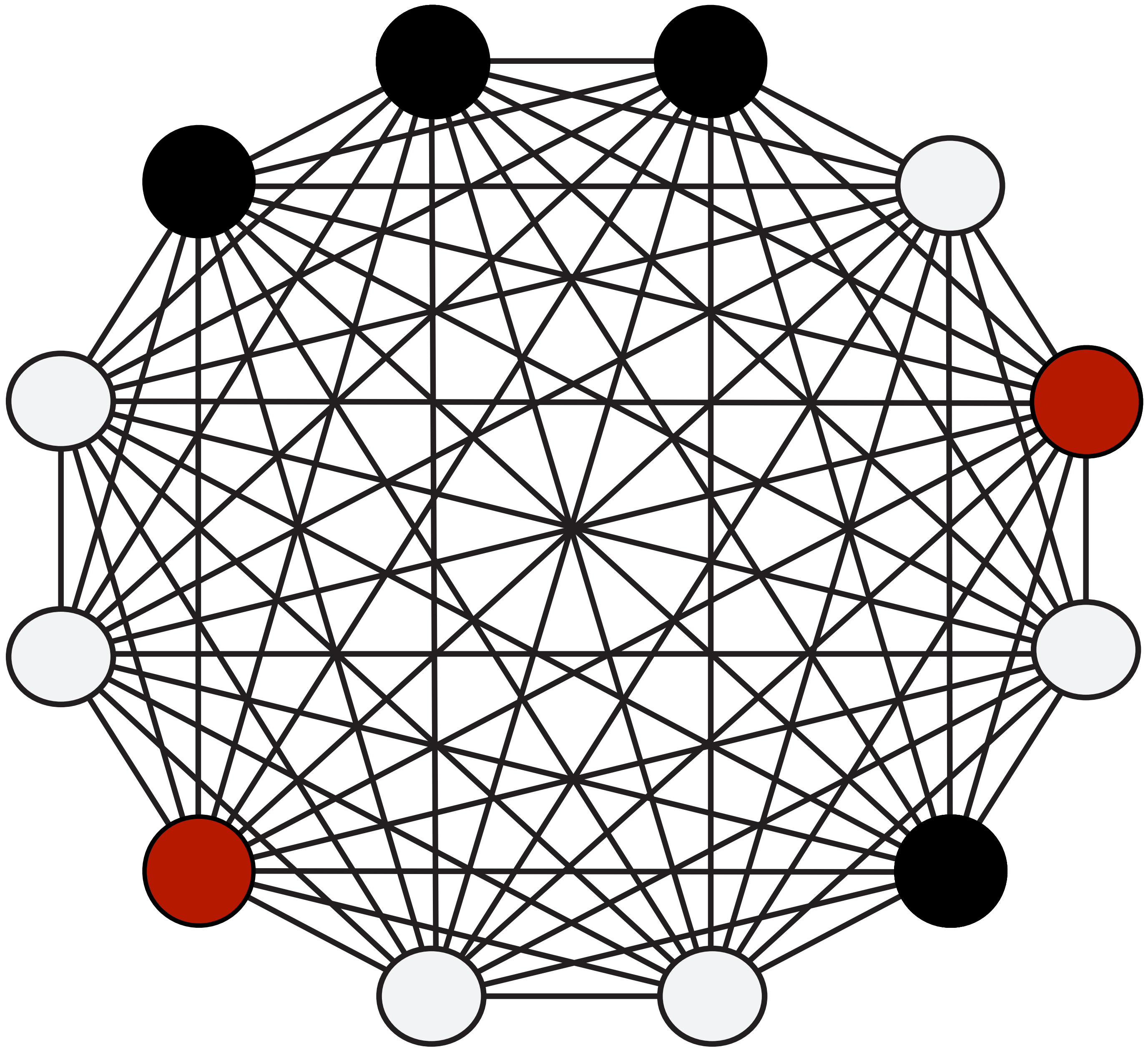

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$$\mathbb{P}(E^N_{ext}) \quad \xrightarrow[N\to\infty]{} \quad \begin{cases} 0 & \text{if} \quad \lambda \in \\ & \text{if} \quad \lambda = \\ 1 & \text{if} \quad \lambda > \quad . \end{cases}$$

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Decoupling using Yule processes

Transition rates

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Hence, at time t, the probability that $\{S, I\} \to \{I, I\}$ happens before $\{R, I\} \to \{R, R\}$ is

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$\leadsto$ We are going to be able to decouple the evolutions of S and R.

Coupling and decoupling via two Yule processes

Yule processes

Definition (Yule process)

In a Yule process $(Y(t))_{t \geqslant 0}$ of parameter λ, starting with one individual, each individual lives a random time distributed according to a $\mathrm{Exp}(\lambda)$ random variable, and at its death gives birth to two individuals

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$\leadsto$ In particular, the intervals between each discontinuity are distributed according to independent $\text{Exp}(\lambda), \text{Exp}(2\lambda), \text{Exp}(3\lambda), \ldots$ random variables.

Coupling with two Yule processes

Let $(\mathcal{R}(t))_{t \geqslant 0}$ be a Yule process of parameter 1, and $(\mathcal{S}_N(t))_{t \geqslant 0}$ a Yule process of parameter λ, time-reversed at its N-th jump.

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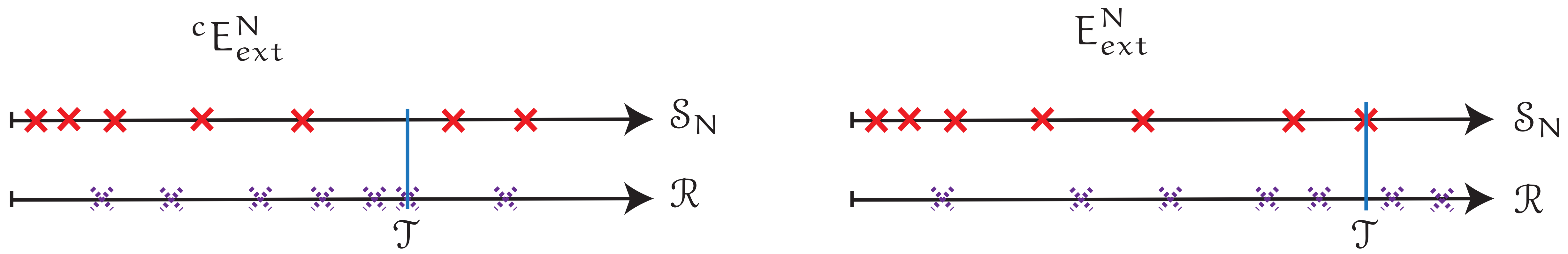

Figure: Ex. $N = 7$, where red crosses represent infections and purple ones recoveries.

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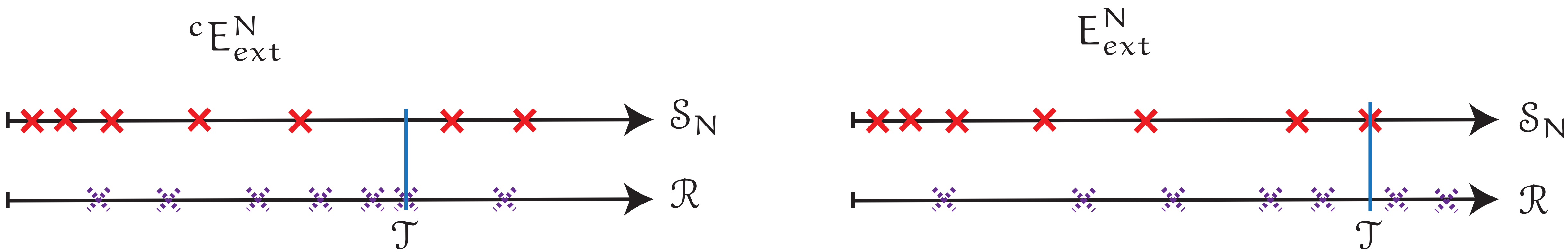

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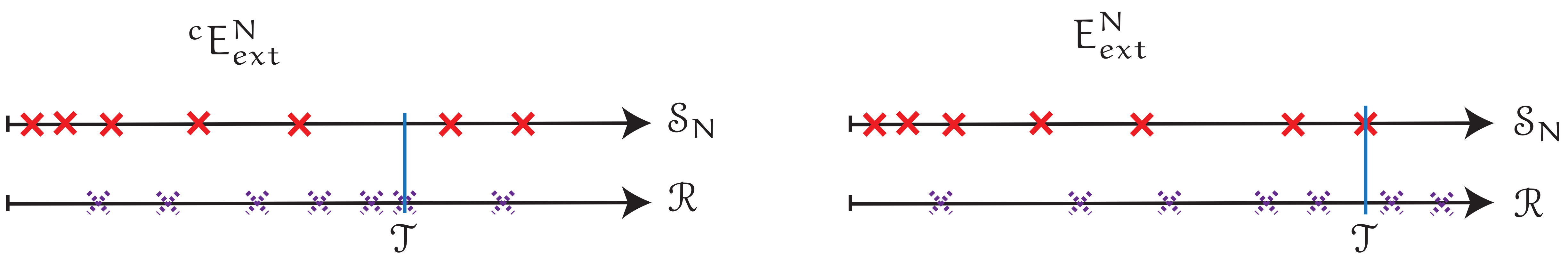

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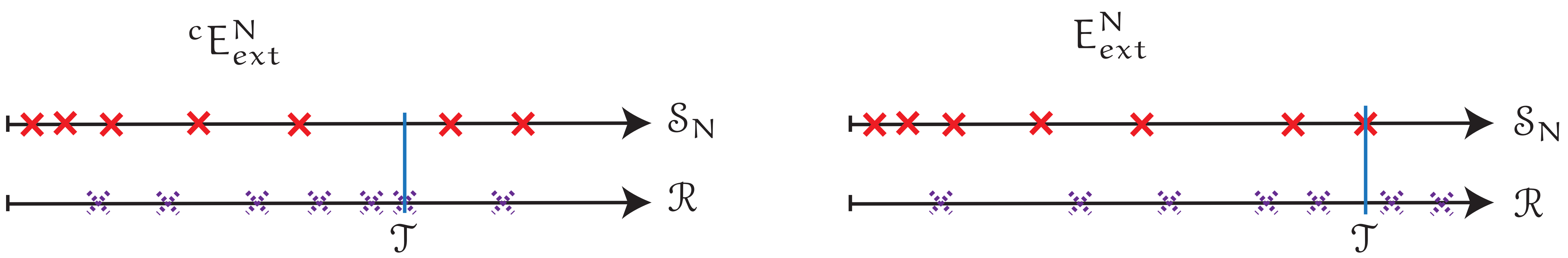

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$\mathcal{T}$ is the smallest between:

- ☞ the first moment when there are more discontinuities of $\mathcal{R}$ than discontinuities of $\mathcal{S}_N$ (I disappears first, ${}^c E_{ext}^N$)
- ☞ the N-th discontinuity of $\mathcal{S}_N$ (S disappears first, E_{ext}^N)

Identification of the critical parameter $\lambda = 1$

Notation.

Denote by $S_N(1), S_N(2), \ldots, S_N(N)$ the discontinuities $\mathcal{S}_N$ and by $R(1), \ldots, R(N)$ the discontinuities of $\mathcal{R}(t)$.

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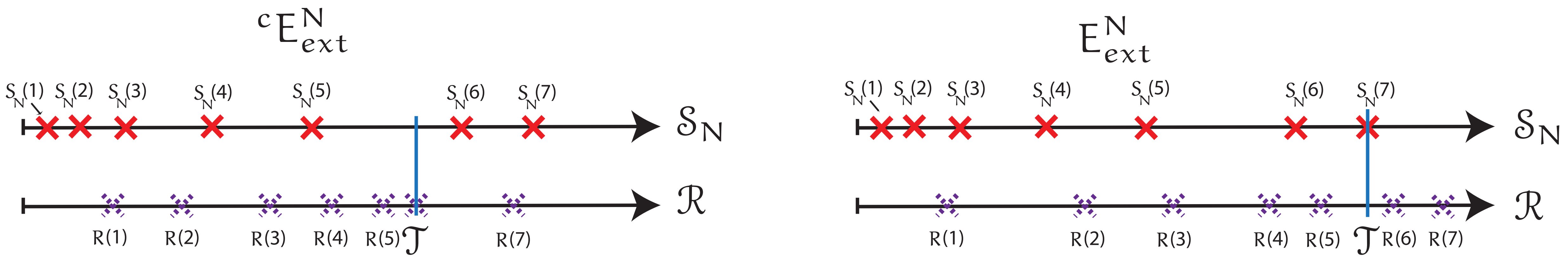

Figure: Example for $N = 7$, where the crosses represent discontinuities.

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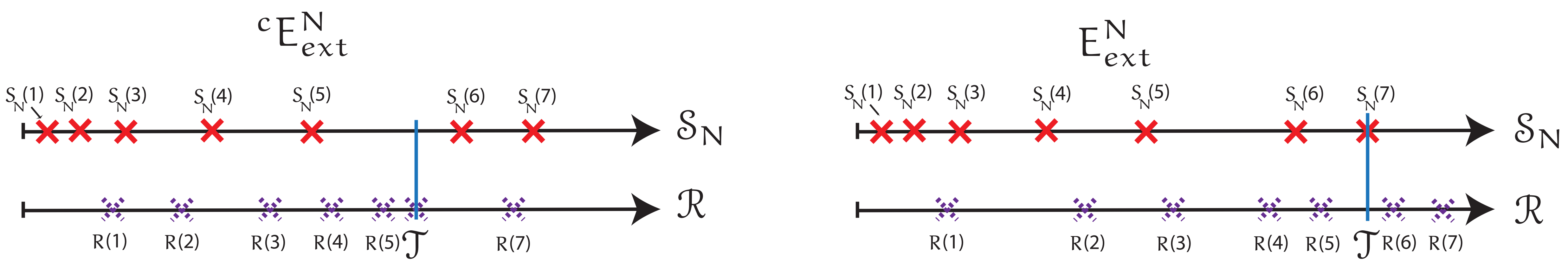

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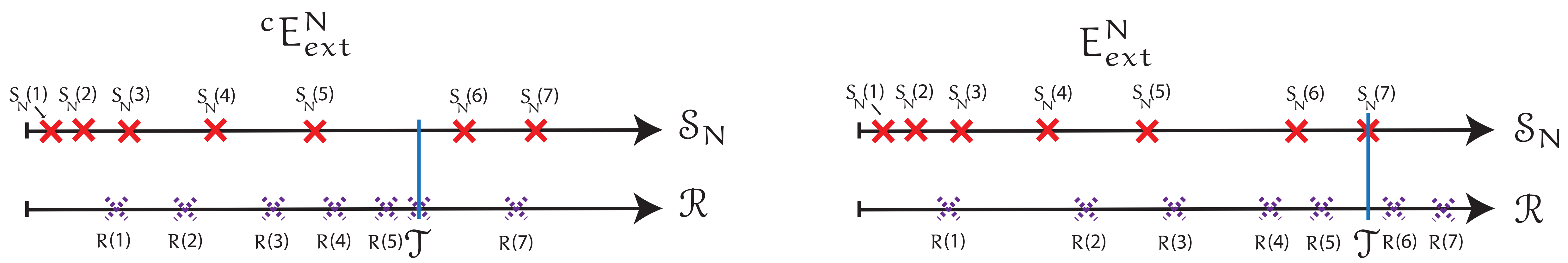

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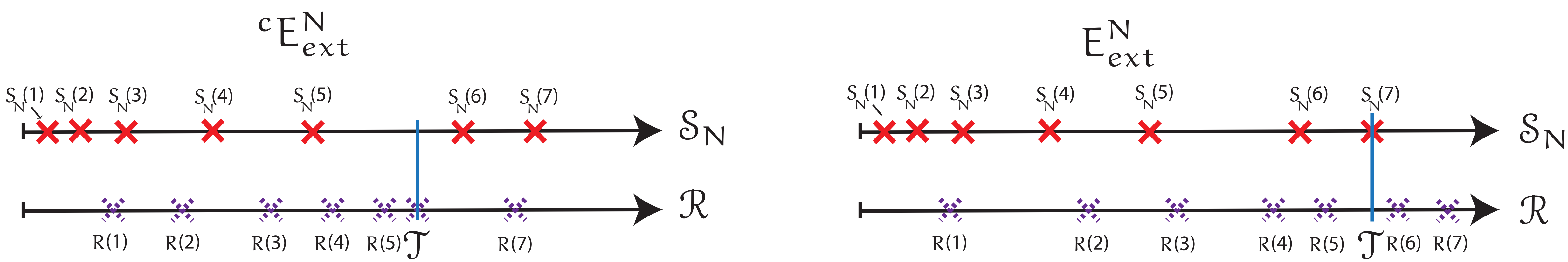

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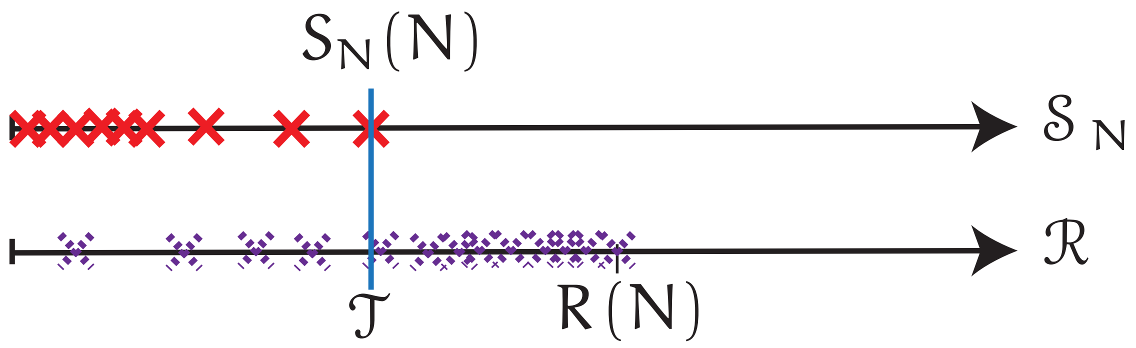

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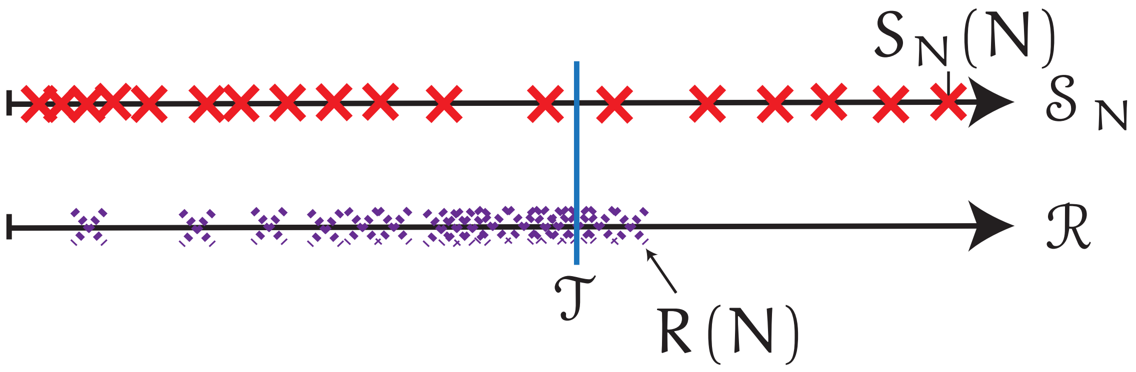

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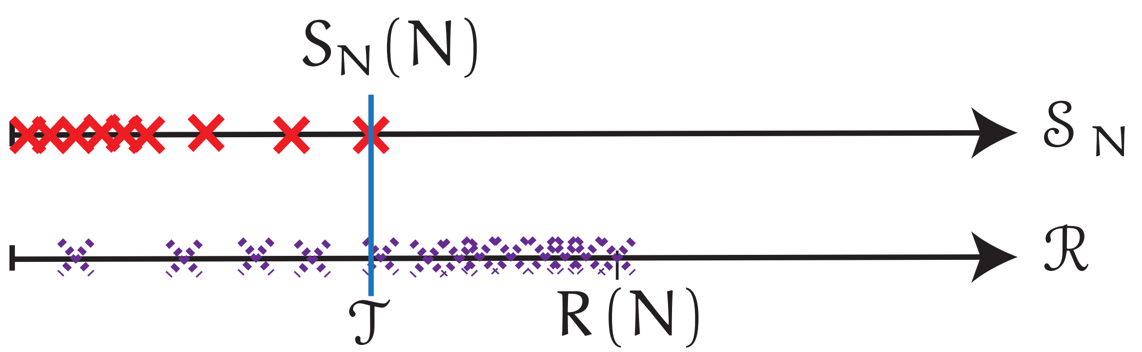

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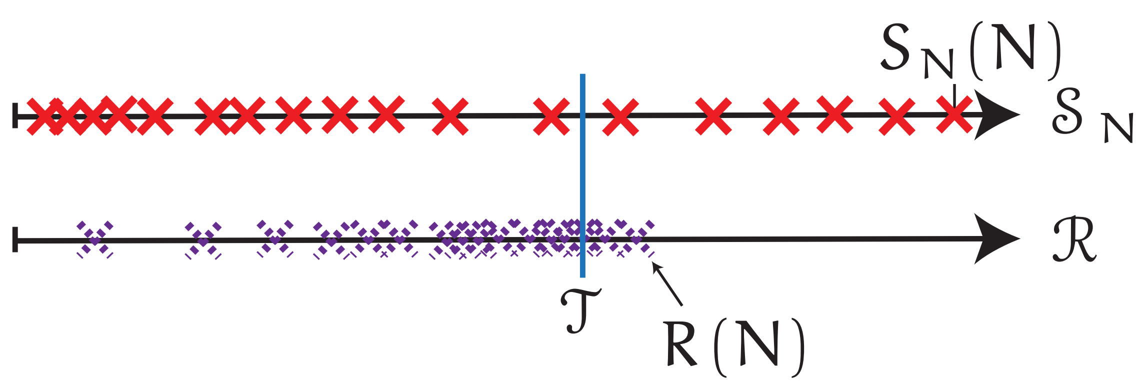

Hence

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Study of the final state of the system

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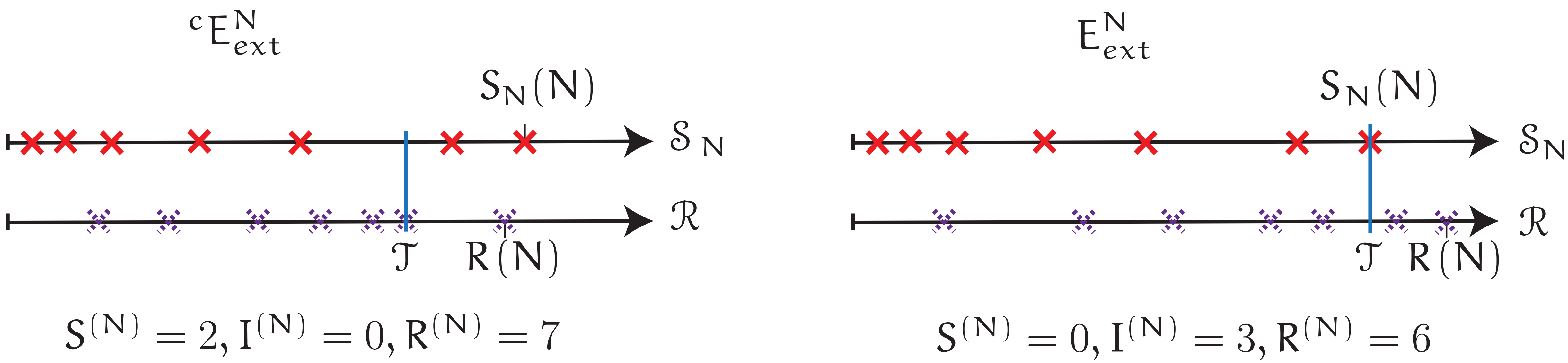

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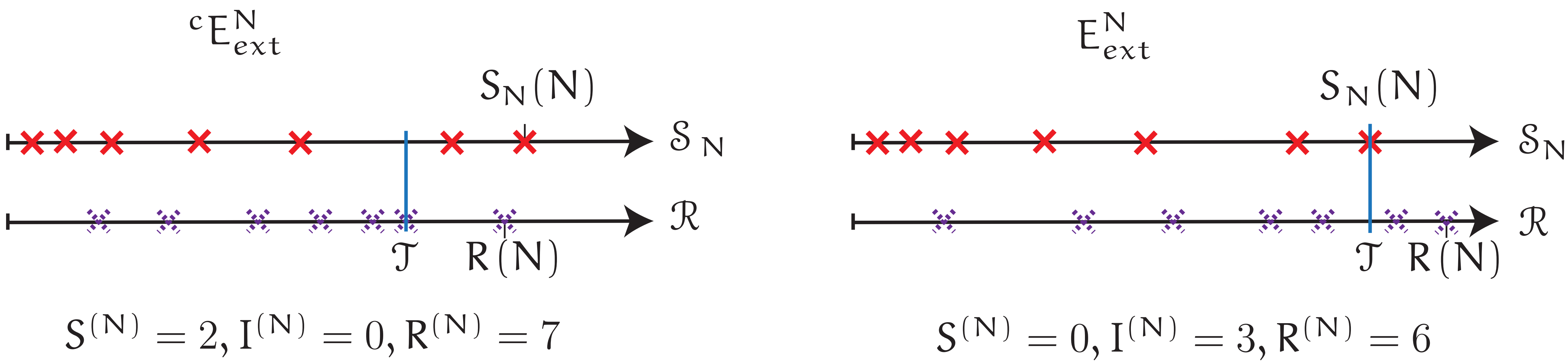

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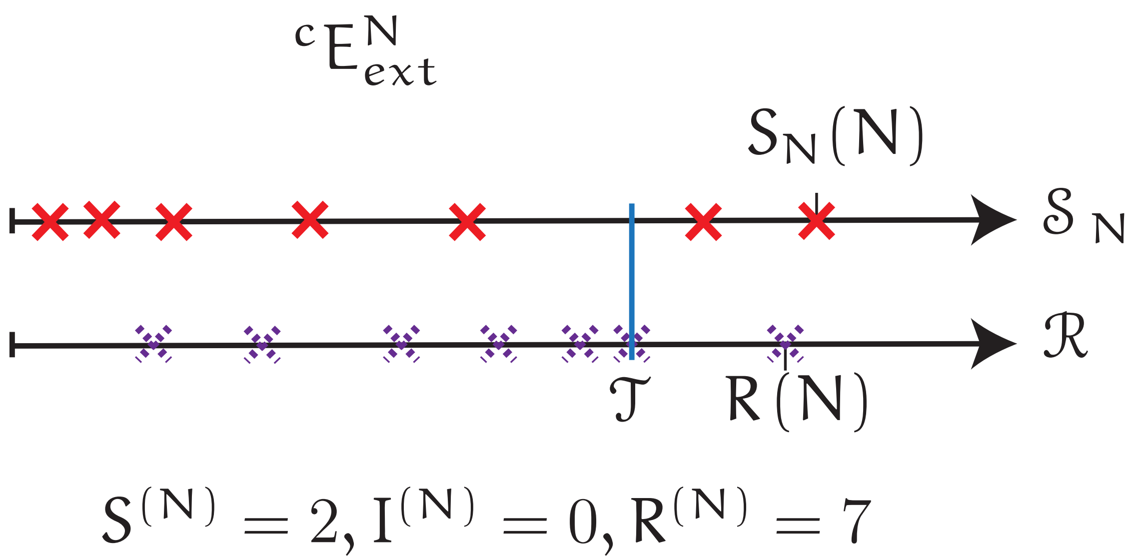

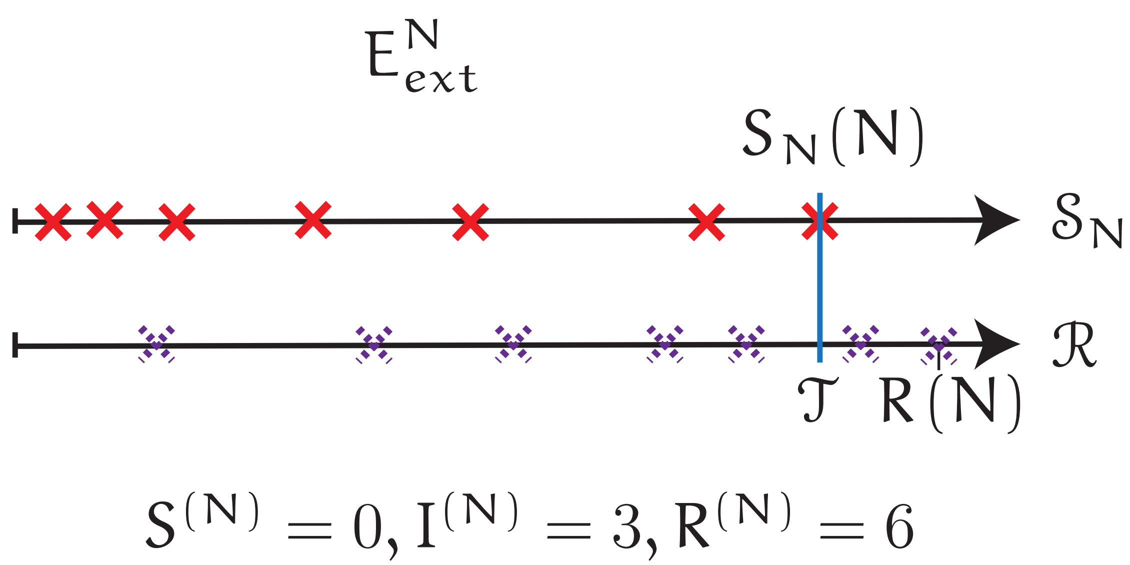

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This should be related to the asymptotic behavior of Yule processes.

Yule processes and terminal value

Proposition

Let $(Y(t))_{t \geqslant 0}$ be a Yule process of parameter λ.

1) We have the convergence

$$e^{-\lambda t} Y_t \quad \xrightarrow[t \to \infty]{a.s.} \quad \mathcal{E},$$

where $\mathcal{E}$ is a $\mathrm{Exp}(1)$ random variable, called *terminal value* of Y.

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Corollary

if τ_N denotes the N-th jump time of Y, then

$$\lambda \tau_N - \ln(N) \quad \xrightarrow[N \to \infty]{a.s.} \quad -\ln(\mathcal{E})$$

Number of susceptible individuals remaining

Theorem (K. '13)**.**

(i) Fix $\lambda \in (0, 1)$.

(ii) Fix $\lambda = 1$.

(iii) Fix $\lambda > 1$.

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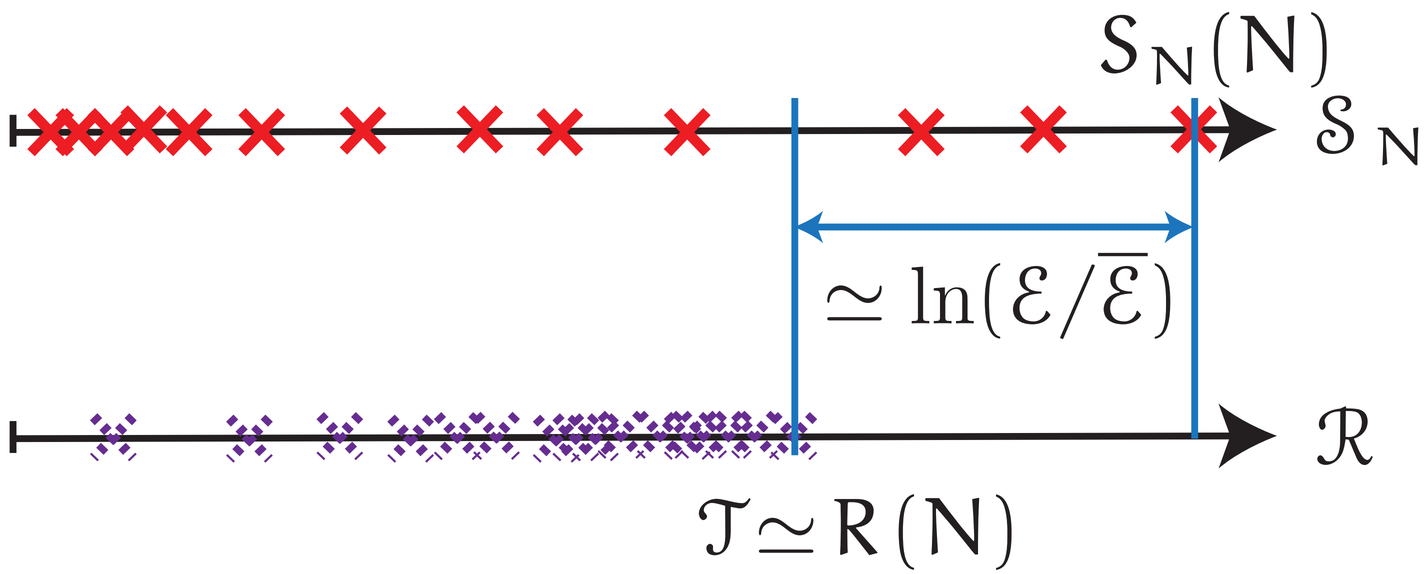
(ii) Fix $\lambda = 1$. Then for every $i \geqslant 0$,

$$\mathbb{P}\left(S^{(N)} = i\right) \xrightarrow[N\to\infty]{} 1/2^{i+1}.$$

(iii) Fix $\lambda > 1$. Then $S^{(N)}$ converges in probability towards 0 as $N \to \infty$.

Idea of the proof: case $\lambda = 1$

On the event ${}^{c}E^{N}_{ext}$,



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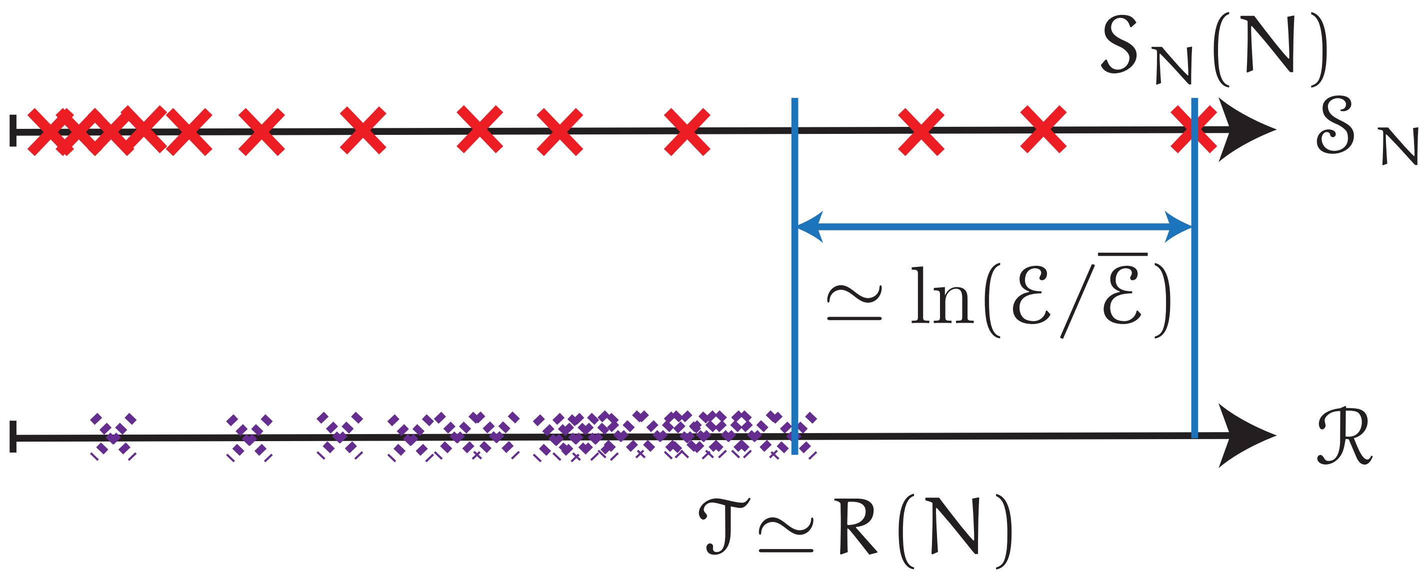

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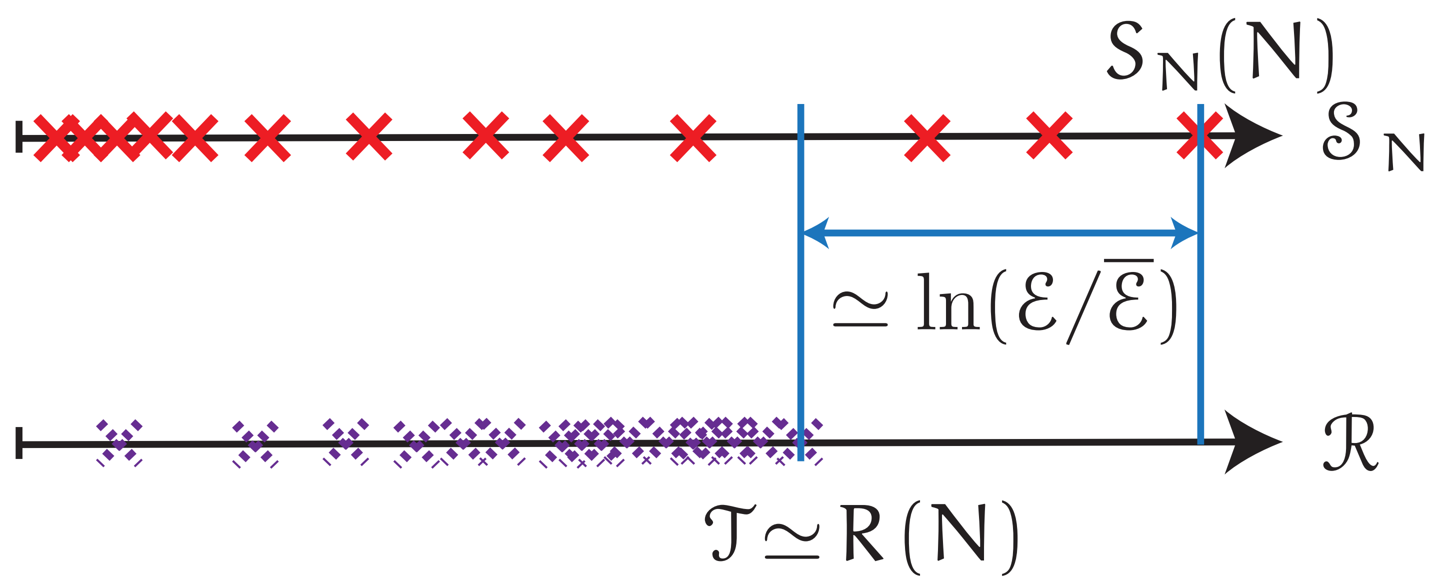

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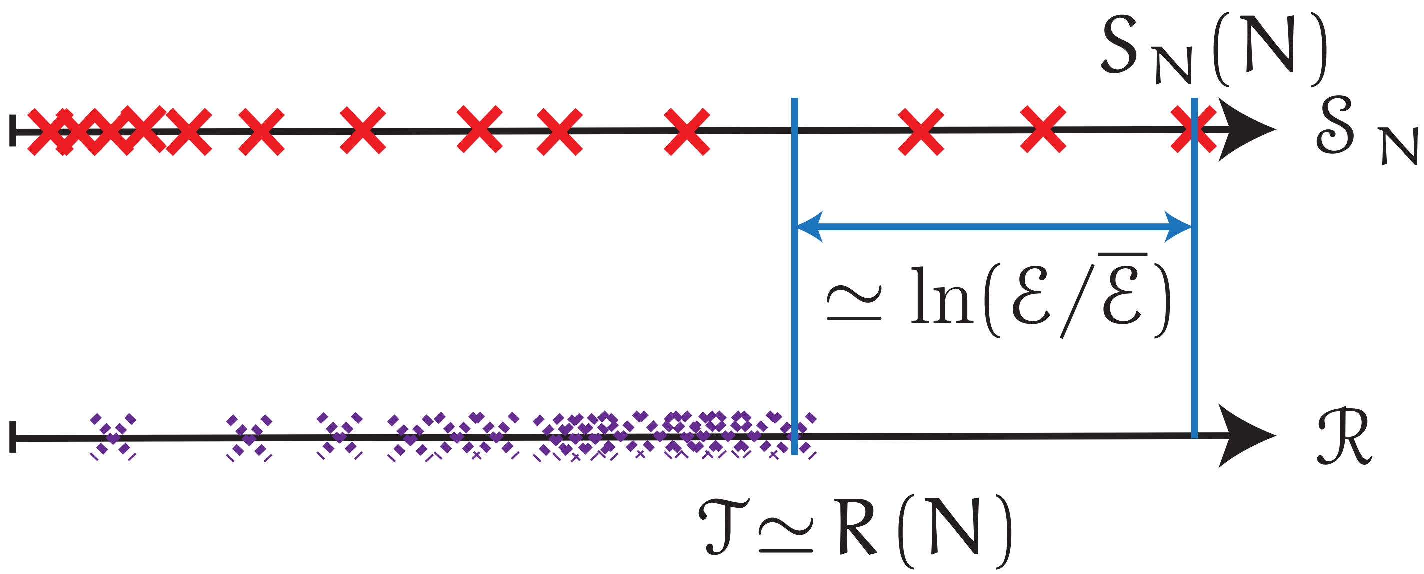

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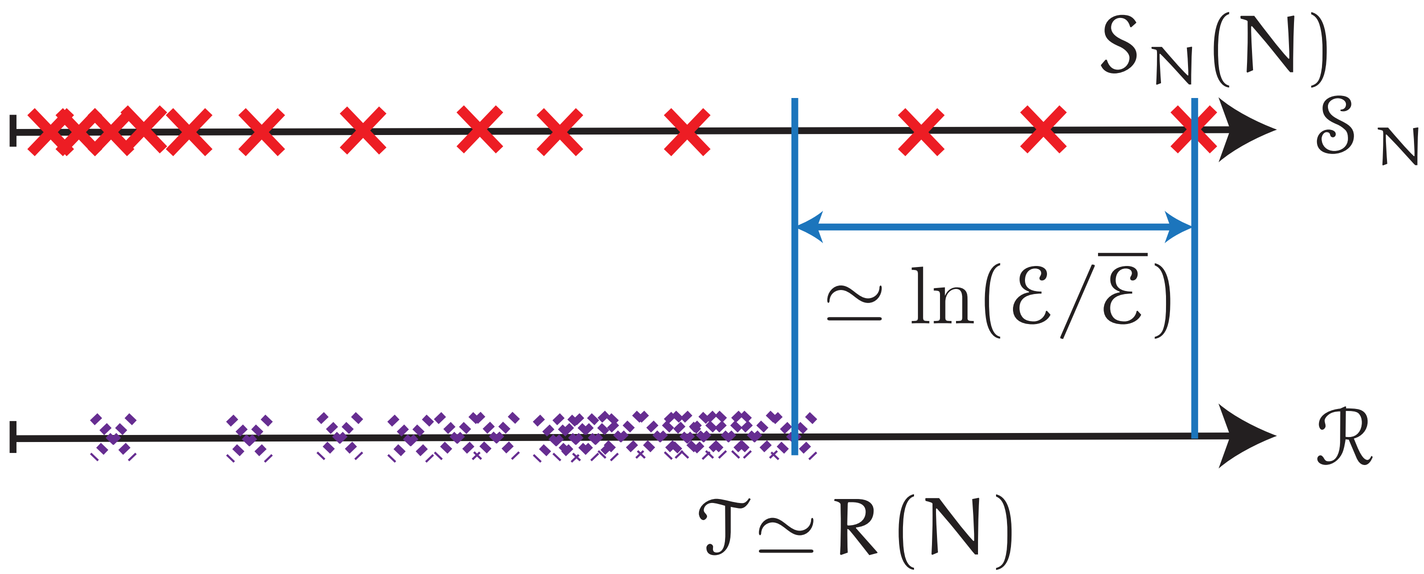

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Thus, $S^{(N)} \simeq$ value of a Yule process of parameter λ at time $\ln(\mathcal{E}/\overline{\mathcal{E}})$, conditionnally on $\mathcal{E}/\overline{\mathcal{E}} > 1$.

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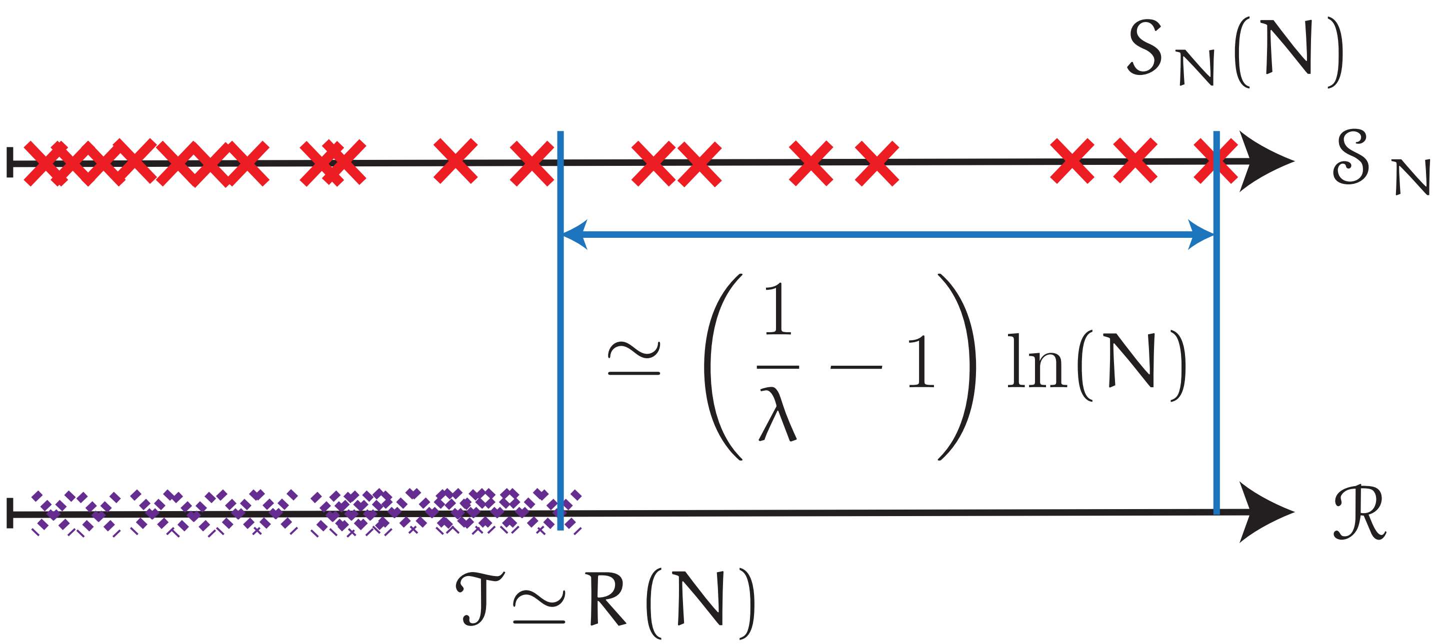

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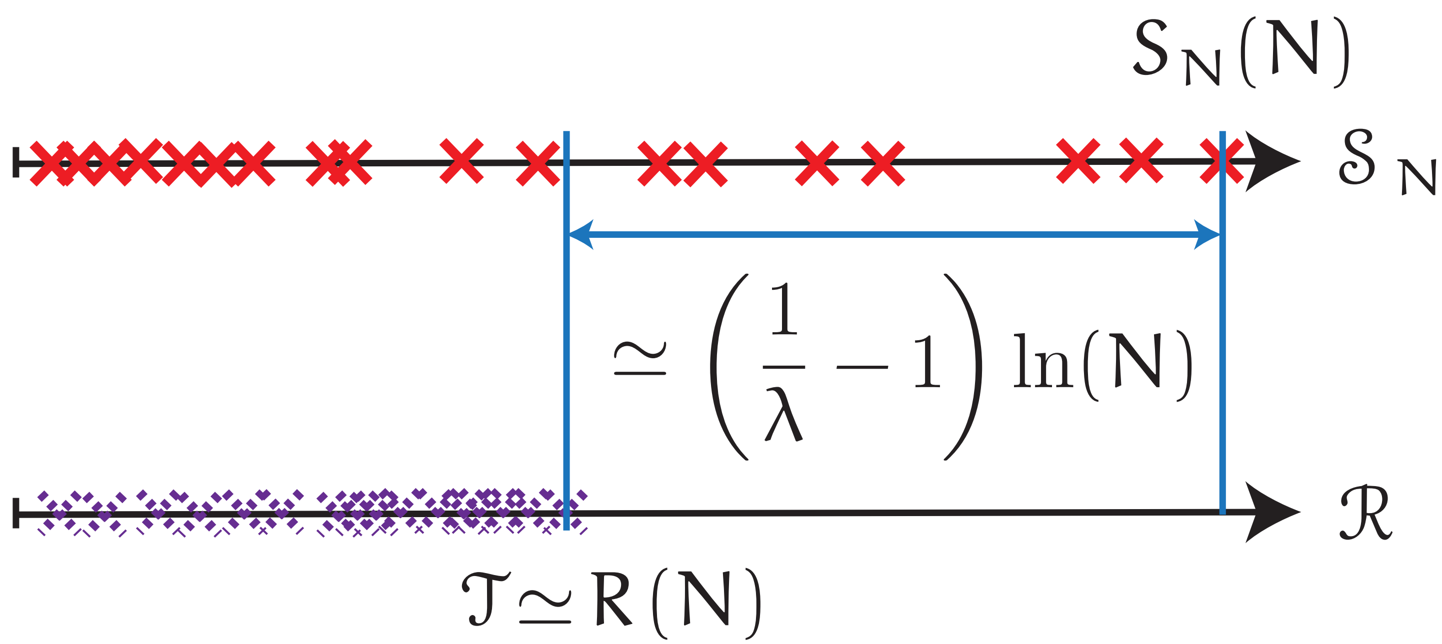

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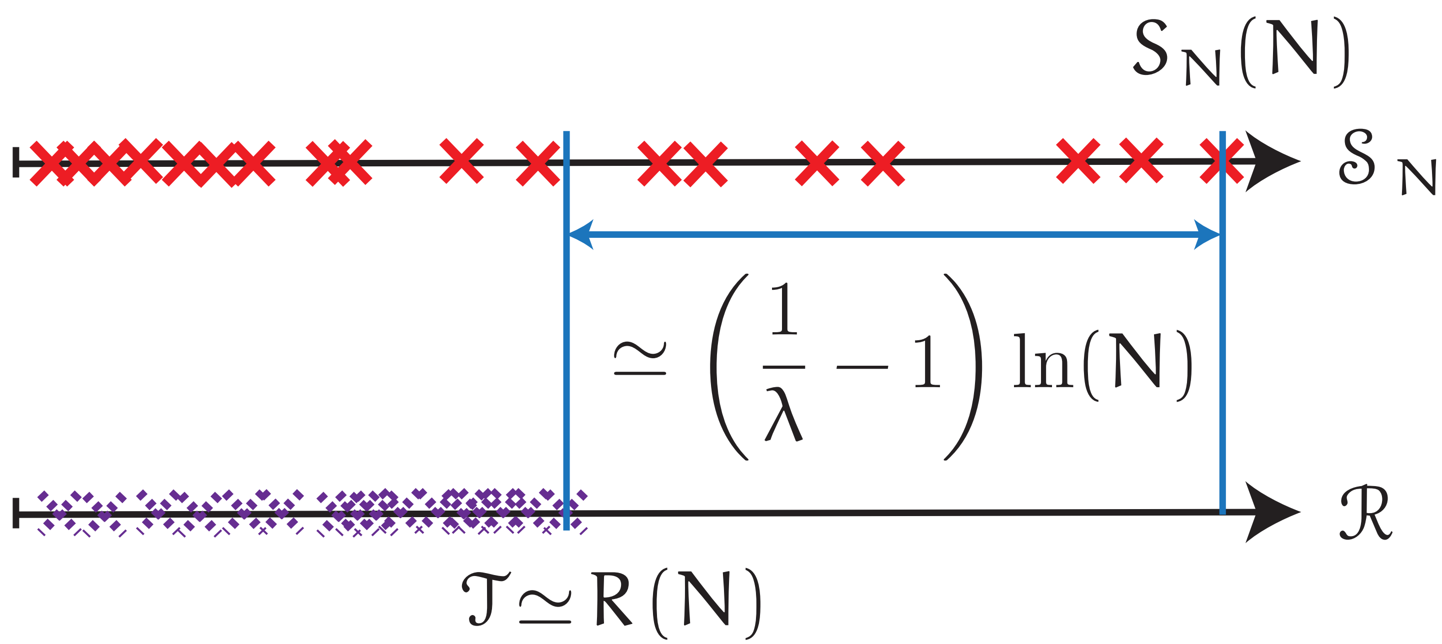

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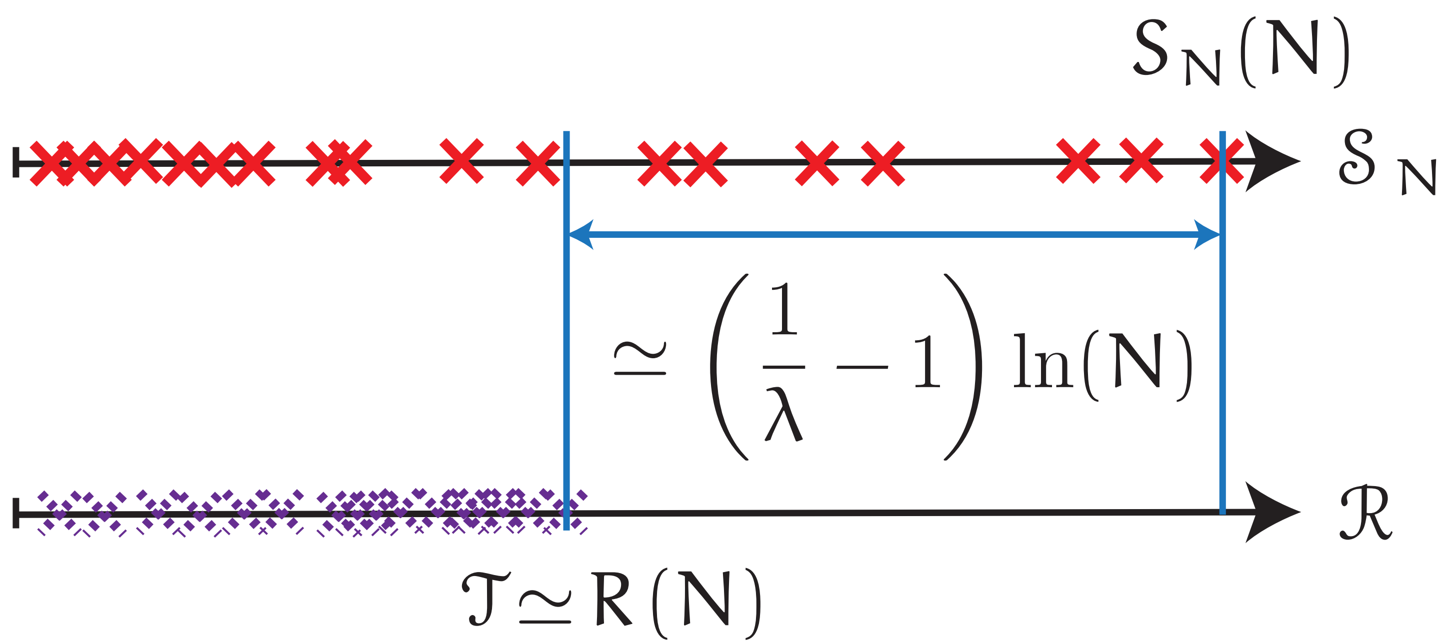

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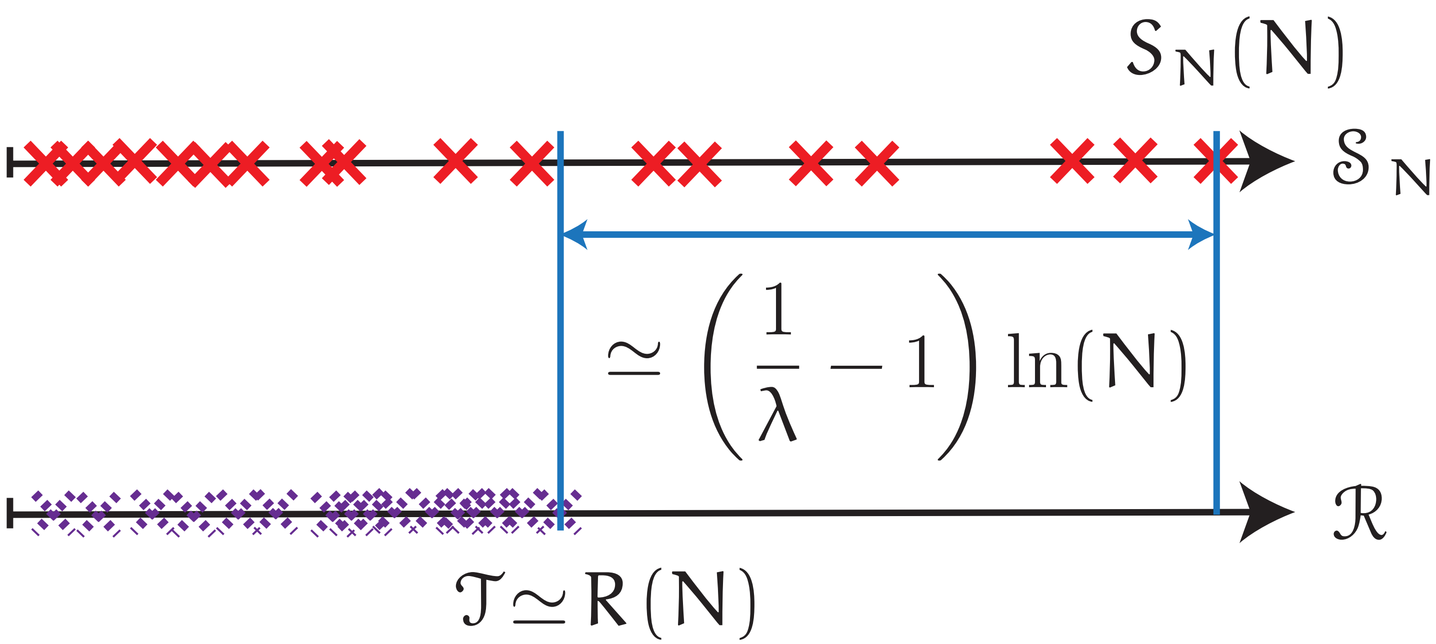
Idea of proof: case $\lambda \in (0,1)$



Recall that $\overline{\mathcal{E}}$ is the terminal value of the Yule process associated with $\mathcal{S}_N$, and $\mathcal{E}$ is the terminal value of $\mathcal{R}$.

We have $S_N(N) \simeq \frac{1}{\lambda}(\ln(N) - \ln(\overline{\mathcal{E}}))$, $R(N) \simeq \ln(N) - \ln(\mathcal{E})$.

Thus, $S^{(N)} \simeq$ value of a Yule process of parameter λ at time $(1/\lambda - 1)\ln(N)$. Which is of order $e^{\lambda(1/\lambda - 1)\ln(N)} = N^{1-\lambda}$.

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Theorem (K. '13).

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(ii) Fix $\lambda = 1$.

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Theorem (Kendall '66)

Let $(\mathcal{P}_t)_{t\geqslant 0}$ *be a Poisson process of parameter* 1 *starting from* 0*, and* $\mathcal{E}$ *be an exponential random variable of parameter* 1*. Then*

$$t \mapsto \mathcal{P}_{\mathcal{E}(e^{\lambda t}-1)} + 1$$

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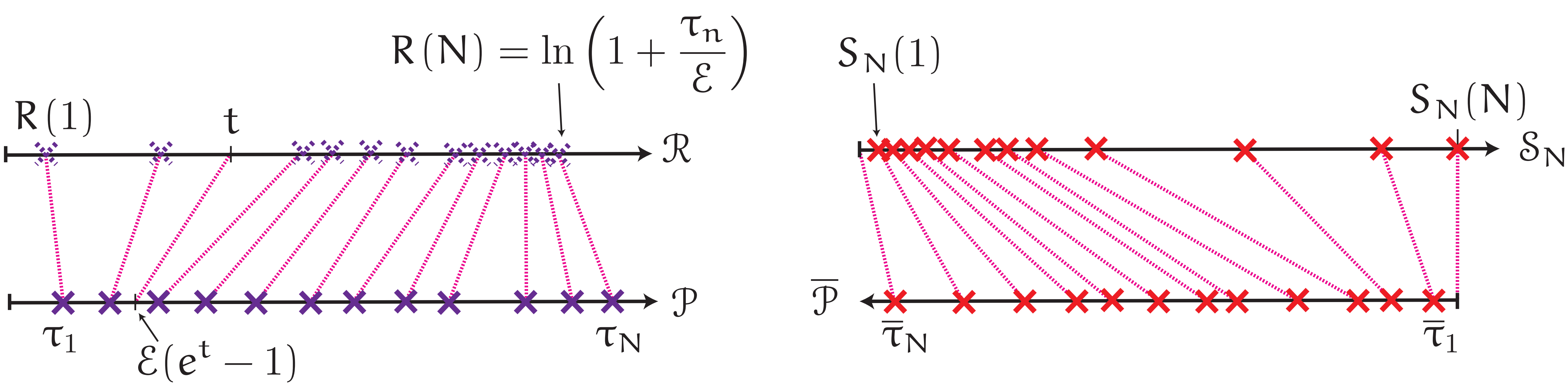

Figure: Illustration of the coupling of Yule processes with Poisson processes

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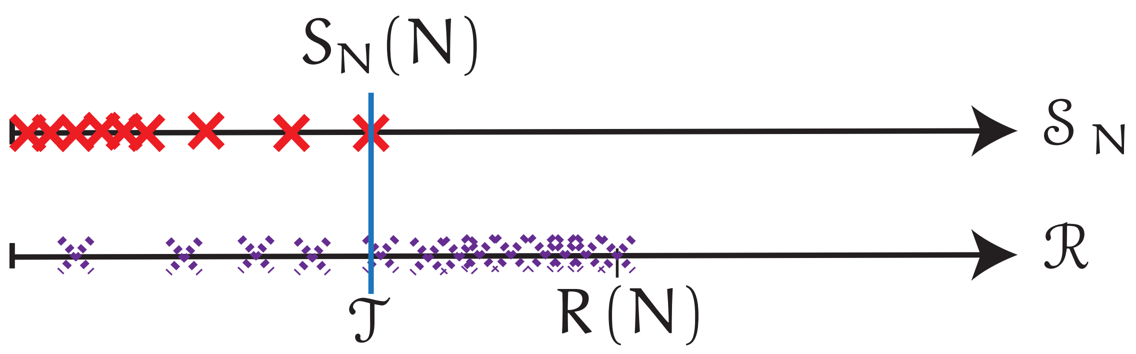
This allows to calculate explicitly the limiting laws in the previous theorems, and to justify the approximation:

A typical situation for $\lambda > 1$:



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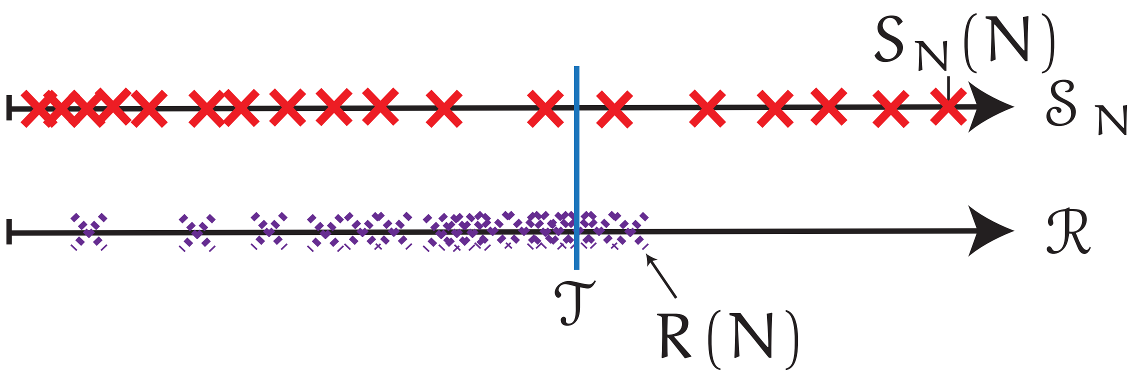


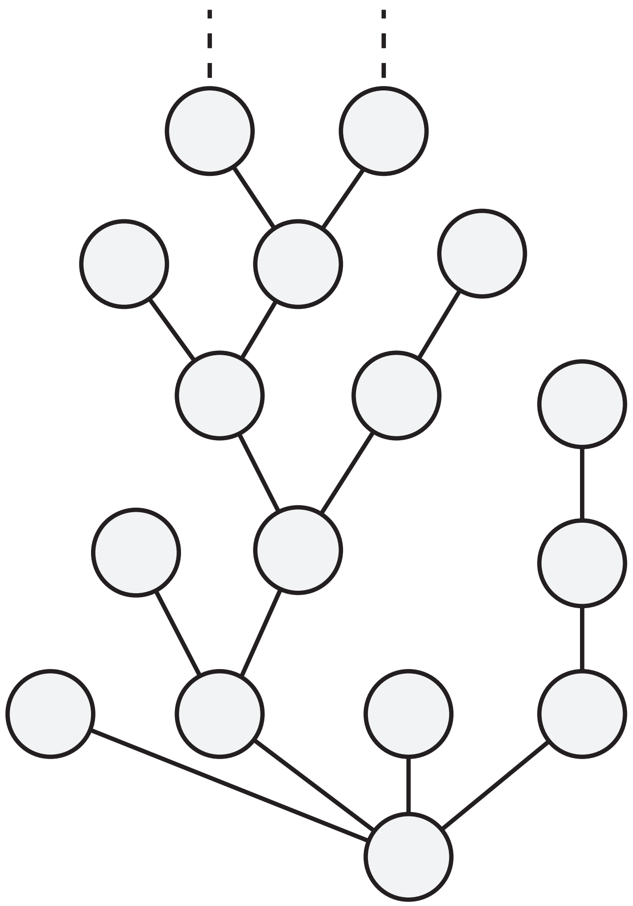
I. Test your intuition!

II. Preys & predators on a complete graph

III. Preys & predators on an infinite tree

Prey-predators on trees

Let T be a rooted tree



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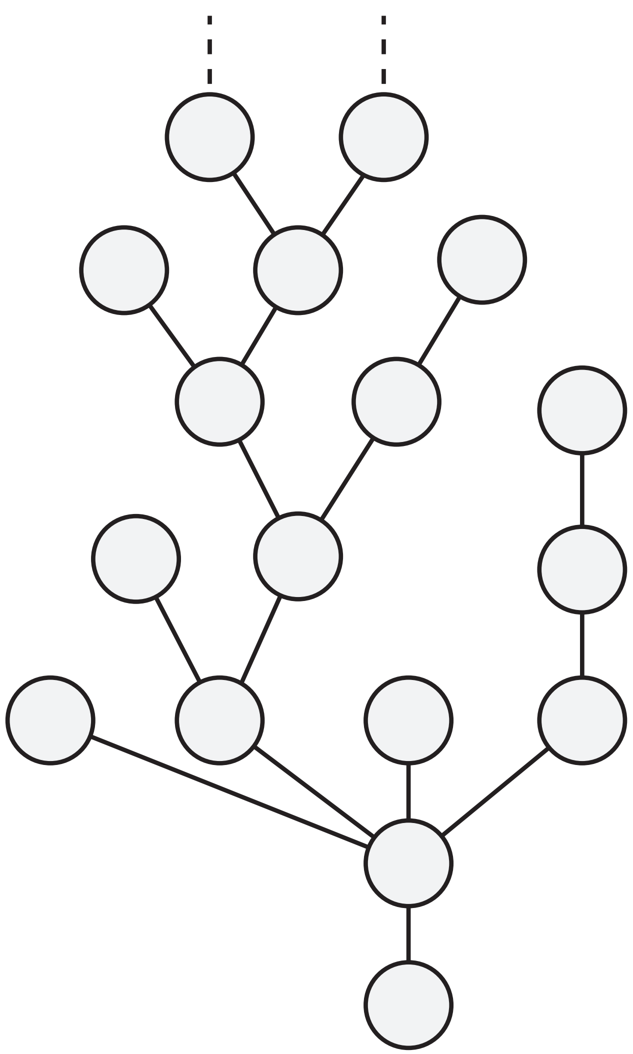

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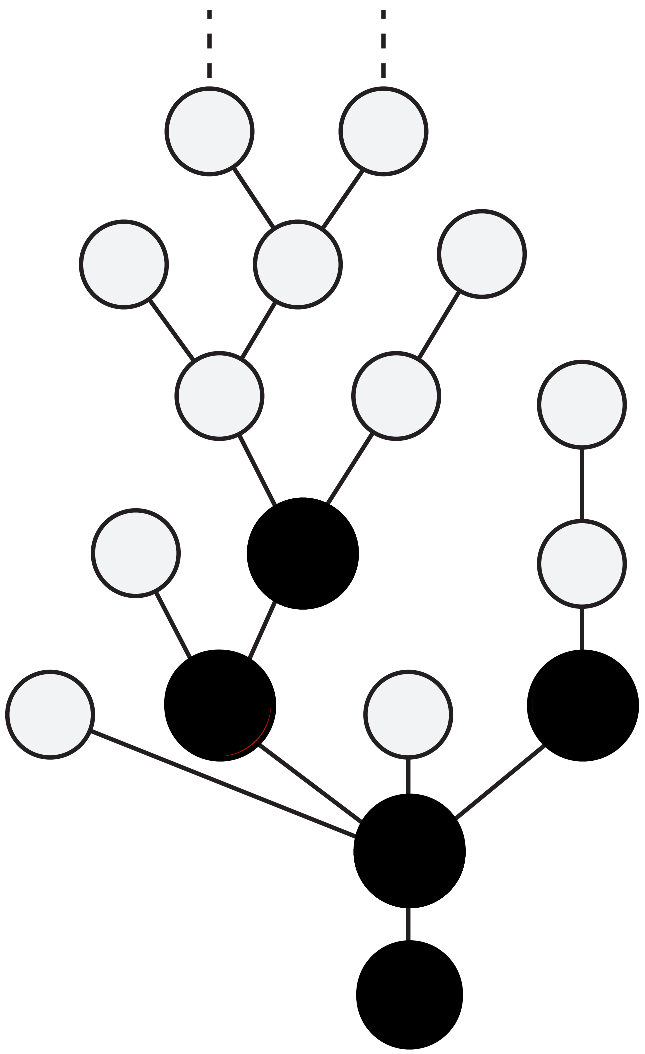

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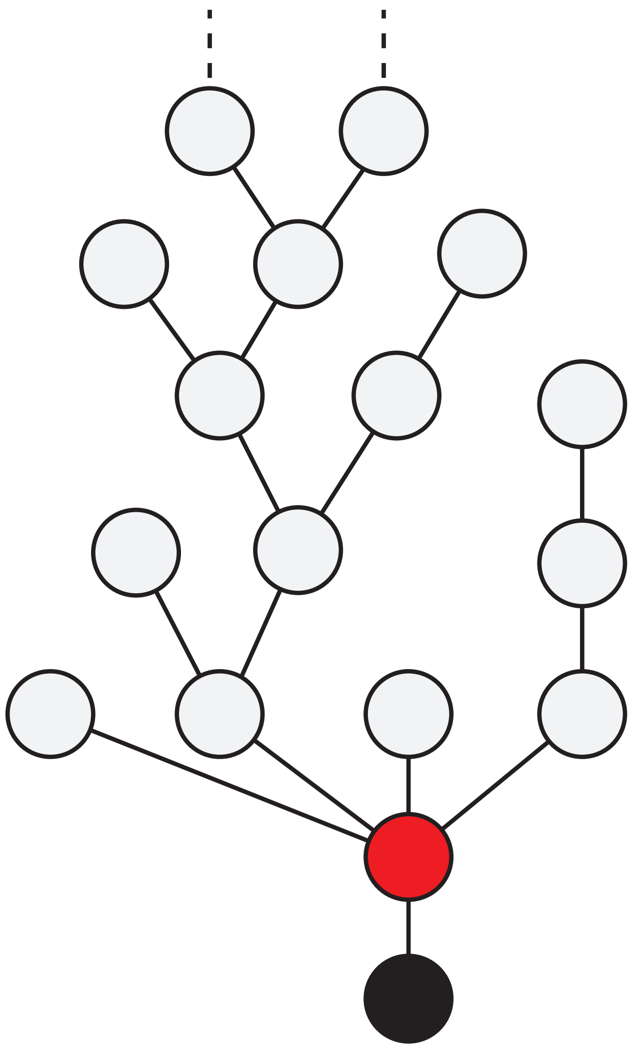

What is the probability $p_T(\lambda)$ that the preys survive indefinitely?

Prey-predators on Galton–Watson trees

Let ν be a probability measure on $\mathbb{Z}_+$. Set $d := \sum_{i \geqslant 0} i\nu(i)$ and assume that $d > 1$. Let $\mathcal{T}$ be a Galton–Watson tree with offspring distribution ν.

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Theorem (Bordenave '12)

If $\lambda < \lambda_c$, we have (under an integrability assumption on ν)

$$\sup\{u \geqslant 1; \mathbb{E}[Z^u] < \infty\} = \frac{(1 - \lambda + \sqrt{\lambda^2 - 2\lambda(2d-1)+1})^2}{4(d-1)\lambda}$$

Tail of the number of Infected individuals

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$$\mathbb{P}(Z > n) \underset{n \to \infty}{\sim}$$

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⇝ Idea: explicit coupling with a branching random walk killed at the origin, and use results of Aïdékon, Hu & Zindy.

Coupling with a branching random walk

Let V be the branching random walk produced with the point process

$$\mathcal{L} = \sum_{i=1}^{U} \delta_{\{\mathcal{E} - \mathsf{Exp}_i(\lambda)\}},$$

starting from 0, where U is a r.v distributed as ν, where $\mathcal{E}$ is an independent Exp(1) r.v and $(\mathsf{Exp}_i(\lambda))_{i \geqslant 1}$ are independent i.i.d. Exp(λ).

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Proposition.

The number Z of infected individuals has the same distribution as

$$\#\{u \in \mathcal{T}; V(v) \geqslant 0, \forall v \in [\![\emptyset, u]\!]\}.$$