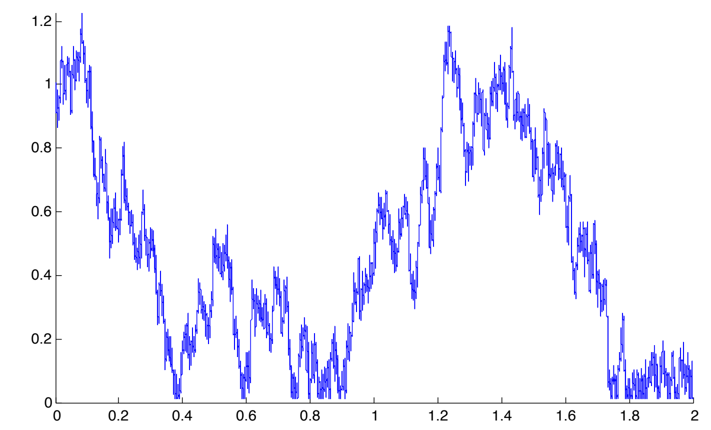
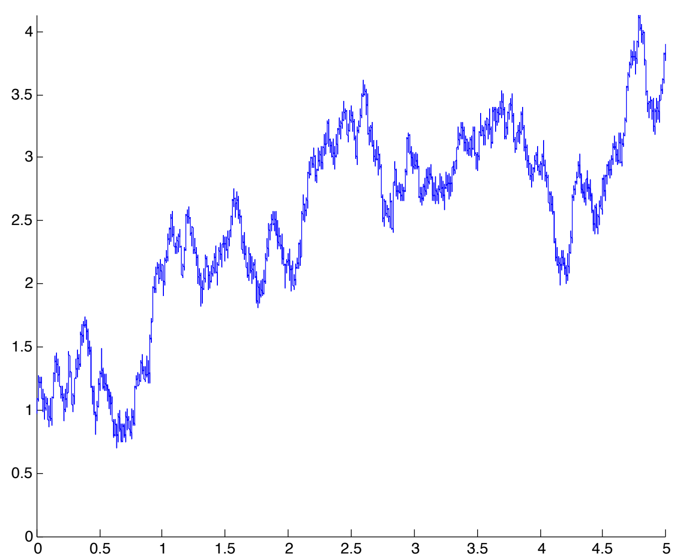
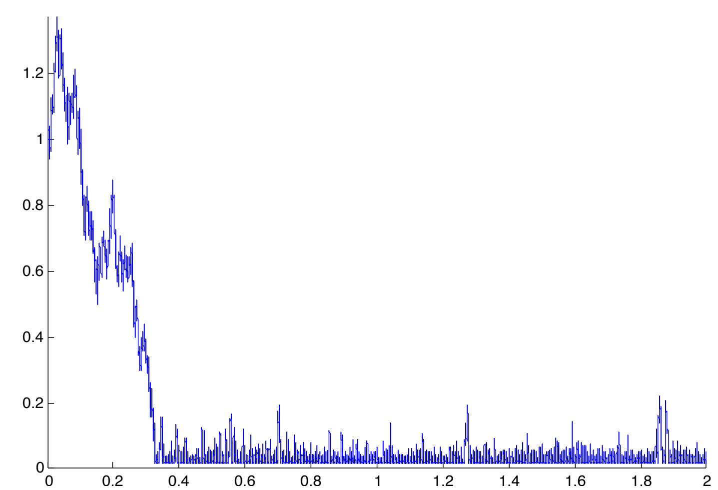
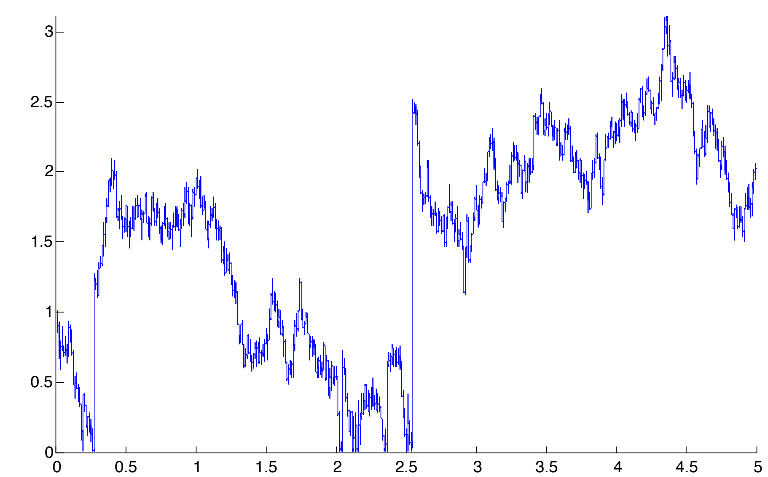
Self-similar scaling limits *of* *Markov chains on the positive integers*









Igor Kortchemski (joint work with Jean Bertoin)
CNRS & CMAP, École polytechnique

Workshop on Lévy processes – Mannheim – May 2015

Outline

I. Goals and motivation

II. Transient case

III. Recurrent case

IV. Positive recurrent case

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5. study separating cycles in large random maps (joint project with Jean Bertoin & Nicolas Curien, which motivated this work)

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holds in distribution (in the space of real-valued càdlàg functions $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ on $\mathbb{R}_+$ equipped with the Skorokhod topology).

Simple example

If $p_{1,2} = 1$ and $p_{n,n\pm1} = \pm\frac{1}{2}$ for $n \geqslant 2$:

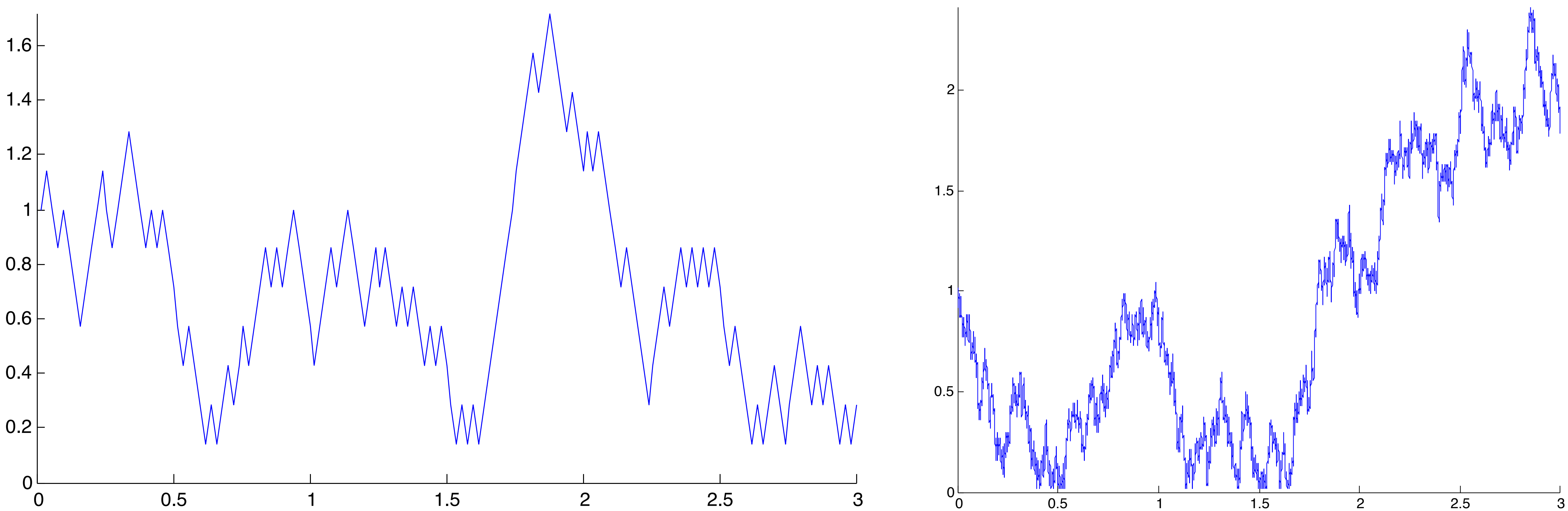

Figure: Linear interpolation of the process $\left(\frac{X_n(\lfloor n^2 t \rfloor)}{n}; 0 \leqslant t \leqslant 3\right)$ for $n = 50$ and $n = 5000$.

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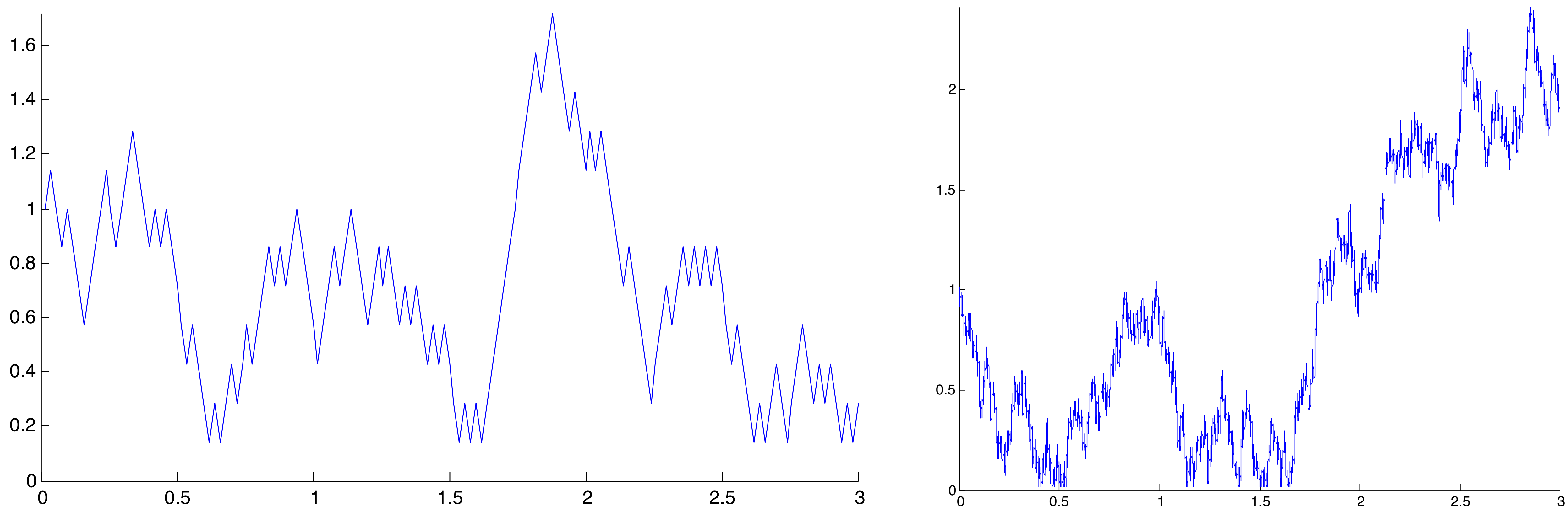

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The scaling limit is reflected Brownian motion.

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$\leadsto$ In the case of Markov chains, one naturally expects the Markov property to be preserved after convergence: the scaling limit should belong to the class of self-similar Markov processes on $[0, \infty)$.

Nonnegative self-similar Markov processes

Let $(\xi(t))_{t \geqslant 0}$ be a Lévy process with characteristic exponent

$$\Phi(\lambda) = -\frac{1}{2}\sigma^2\lambda^2 + ib\lambda + \int_{-\infty}^{\infty} \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{|x| \leqslant 1}\right) \Pi(dx), \qquad \lambda \in \mathbb{R}$$

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- $I_\infty = \infty$ a.s. if ξ drifts to $+\infty$ or oscillates.

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We will write that Y is a $\mathrm{pSSMP}_1^{(\gamma)}(\sigma, b, \Pi)$.

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Let Π be a measure on $\mathbb{R}\backslash\{0\}$ such that

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\ \Pi(dx) < \infty.$$

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V. Applications

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This means that

$$a_n \cdot \mathbb{E}\left[f\left(\frac{X_n(1)}{n}\right)\right] \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} f(e^x)\, \Pi(dx)$$

for every continuous function f with compact support in $[0, \infty] \backslash \{1\}$, i.e. a jump of the process X_n/n from 1 to x occurs with a small rate $\frac{1}{a_n} \exp \circ \Pi(dx)$.

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$$a_n \cdot \int_{-1}^{1} x \, \Pi_n^*(dx) \xrightarrow[n\to\infty]{} b, \quad a_n \cdot \int_{-1}^{1} x^2 \, \Pi_n^*(dx) \xrightarrow[n\to\infty]{} \sigma^2 + \int_{-1}^{1} x^2 \, \Pi(dx)$$

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(Conditions very close to those giving convergence of infinitely divisible distributions)

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Theorem (Bertoin & K. '14 — transient case)**.**

Assume that **(A1)** and **(A2)** hold, and that $\xi \not\to -\infty$. Then

$$\left(\frac{X_n(\lfloor a_n t \rfloor)}{n}; t \geqslant 0 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad (Y(t); t \geqslant 0)$$

holds in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where Y is a $\text{pSSMP}_1^{(\gamma)}(\sigma, b, \Pi)$.

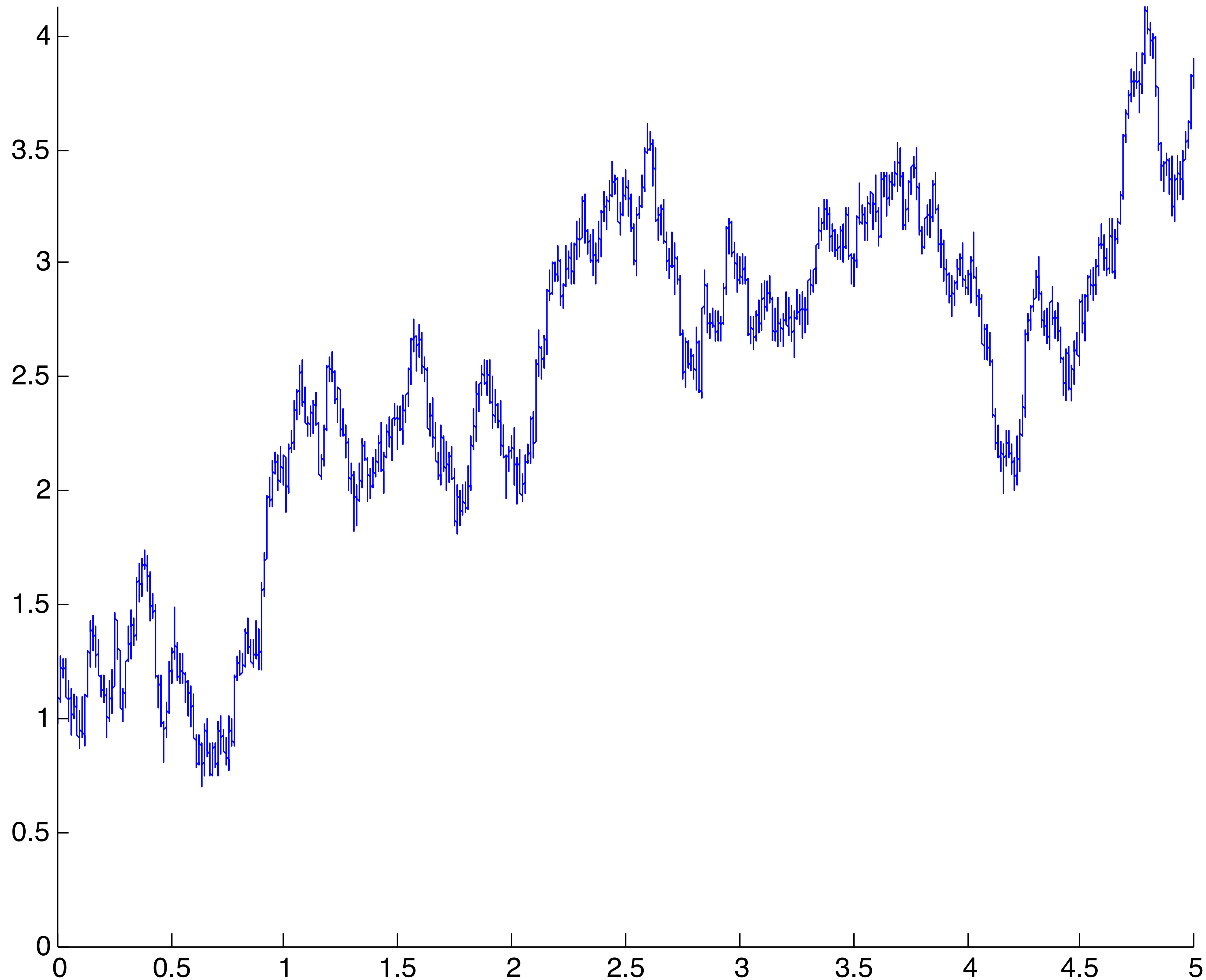

Figure: Illustration of the transient case.

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$$\left(\frac{X_n(\lfloor a_n t \rfloor)}{n}; t \geqslant 0\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (\exp(\xi(\tau(t))); t \geqslant 0) \quad = \quad Y$$

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What happens when ξ drifts to $-\infty$, in which case $I_\infty < \infty$ and Y is absorbed in 0 ?

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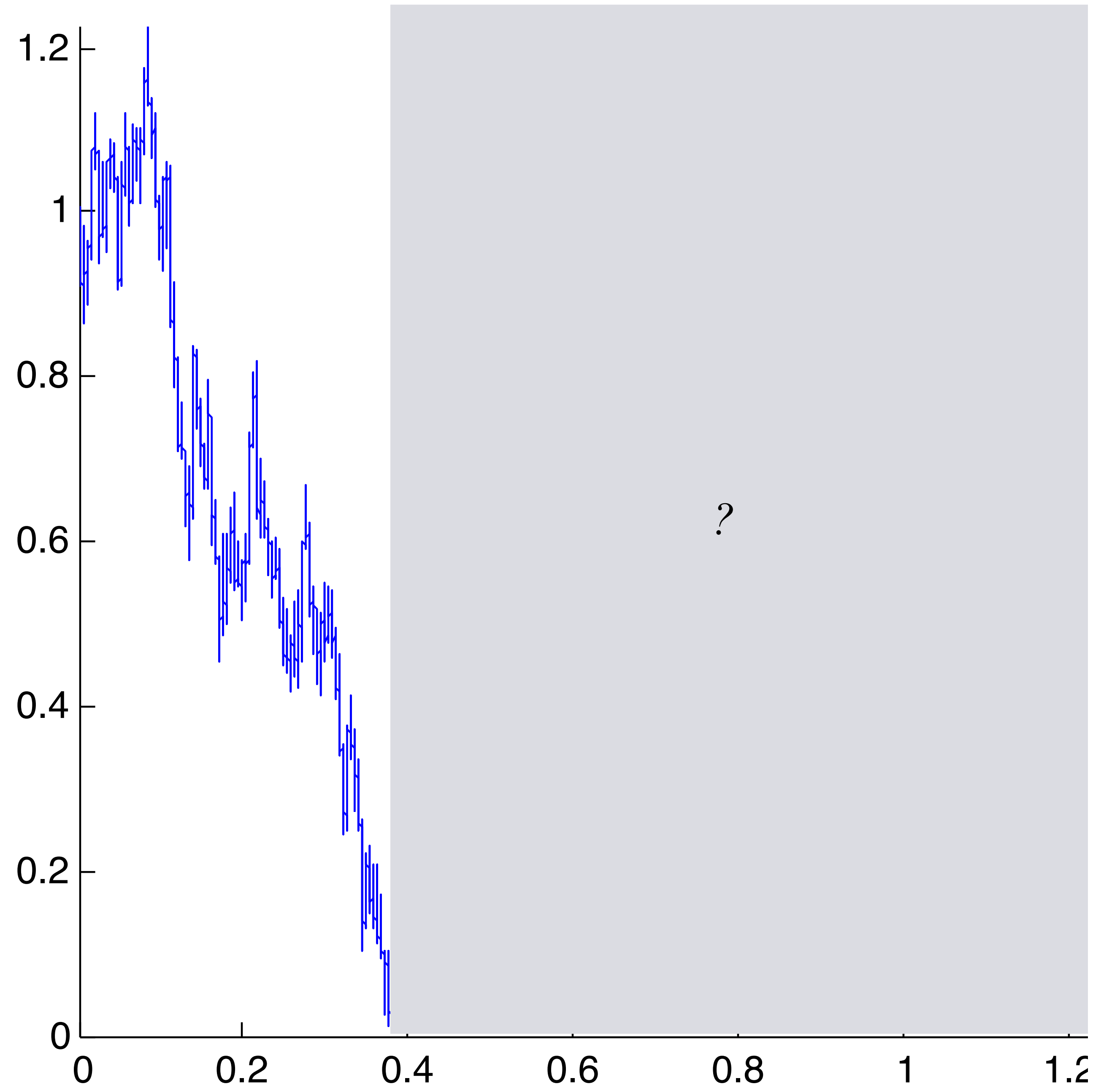

Figure: Illustration of the recurrent case.

Proof of the recurrent case

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↝ In particular, if **(A1)**, **(A2)**, **(A3)** hold and $\xi \to -\infty$ almost surely, $A_i^{(K)} < \infty$ for every $i \geqslant 1$.

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Assume that **(A1)**, **(A2)**, **(A3)** hold and that the Lévy process ξ drifts to $-\infty$. Then

$$\frac{A_n^{(K)}}{a_n} \quad \xrightarrow[n\to\infty]{(d)} \quad \int_0^\infty e^{\gamma \xi(s)} ds.$$

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I. Goals and motivation

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IV. Positive recurrent case

Let Ψ be the Laplace exponent of ξ:

$$\Psi(\lambda) = \Phi(-i\lambda) = \frac{1}{2}\sigma^2\lambda^2 + b\lambda + \int_{-\infty}^{\infty} \left(e^{\lambda x} - 1 - \lambda x \mathbb{1}_{|x|\leqslant 1}\right) \Pi(dx),$$

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so that

$$\mathbb{E}\left[e^{\lambda \xi(t)}\right] = e^{t\Psi(\lambda)}.$$

(A4). There exists $\beta_0 > \gamma$ s.t.

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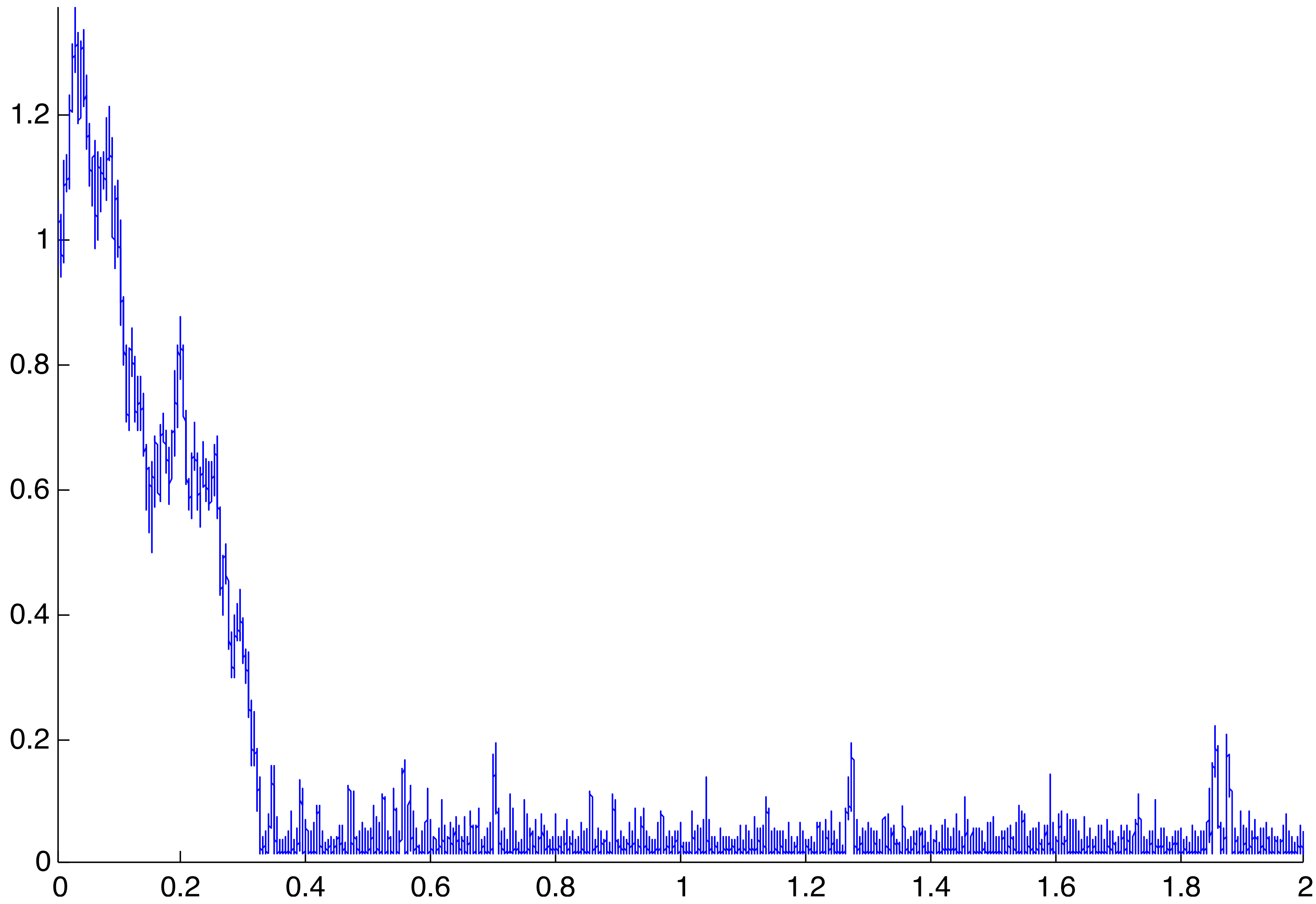

Figure: Illustration of the positive recurrent case.

Foster-Lyapounov is back

- First step: show that

$$\frac{\mathbb{E}\left[A_n^{(K)}\right]}{a_n} \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}\left[\int_0^\infty e^{\gamma\xi(s)}ds\right] = \frac{1}{|\Psi(\gamma)|}.$$

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- Second step: show that that this implies that the maximum of a_n excursions starting from $\{1, 2, \ldots, K\}$ cannot be of order n.

Questions

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Is it true that the "recurrent" case remains valid if **(A3)** is replaced with the condition $\inf\{i \geqslant 1; X_n(i) \leqslant K\} < \infty$ almost surely for every $n \geqslant 1$?

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Assume that **(A1)** , **(A2)** **(A3)** hold, and that there exists an integer $1 \leqslant n \leqslant K$ such that $\mathbb{E}\left[\inf\{i \geqslant 1; X_n(i) \leqslant K\}\right] = \infty$. Under what conditions on the probability distributions $X_1(1), X_2(1), \ldots, X_K(1)$ does the Markov chain X_n have a continuous scaling limit (in which case 0 is a continuously reflecting boundary)? A discontinuous càdlàg scaling limit (in which case 0 is a discontinuously reflecting boundary)?

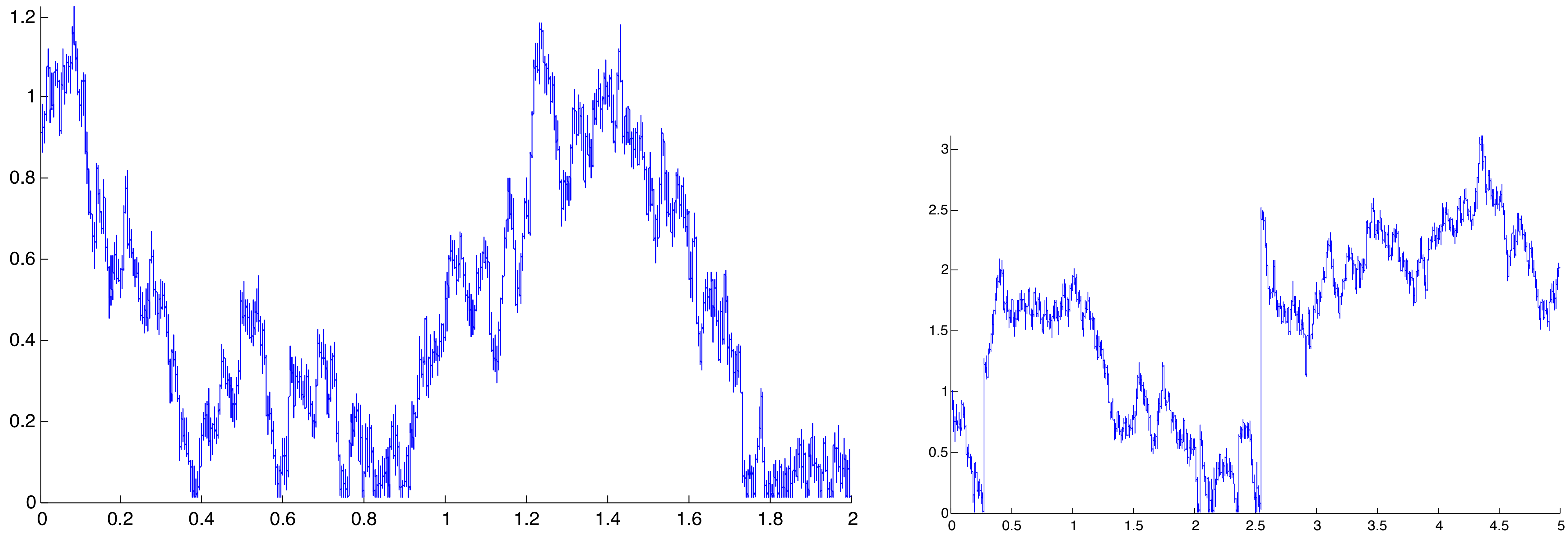

Figure: Illustration of the null recurrent case with different behavior near the boundary.