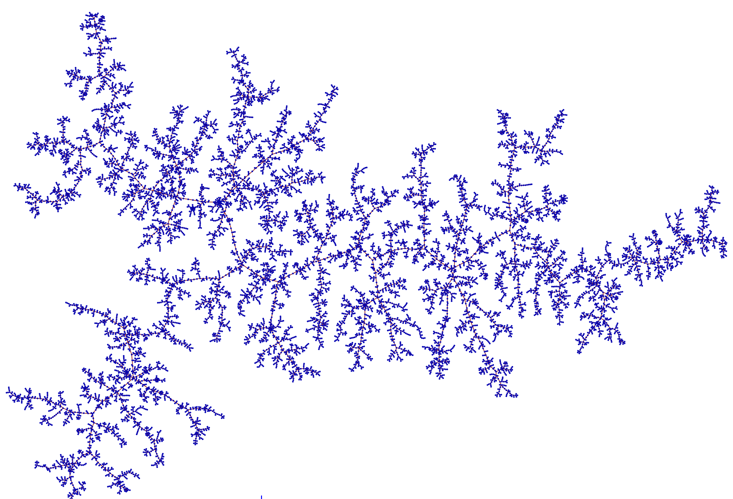
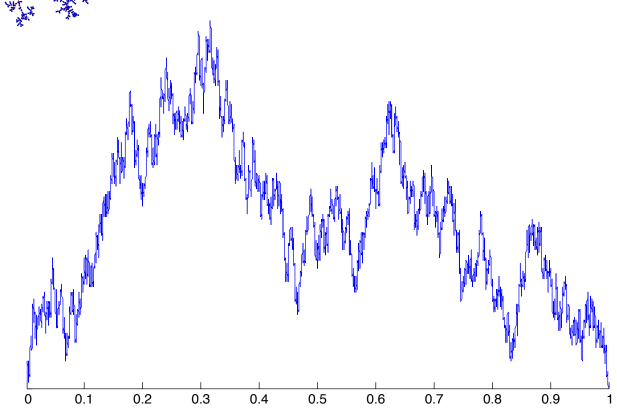
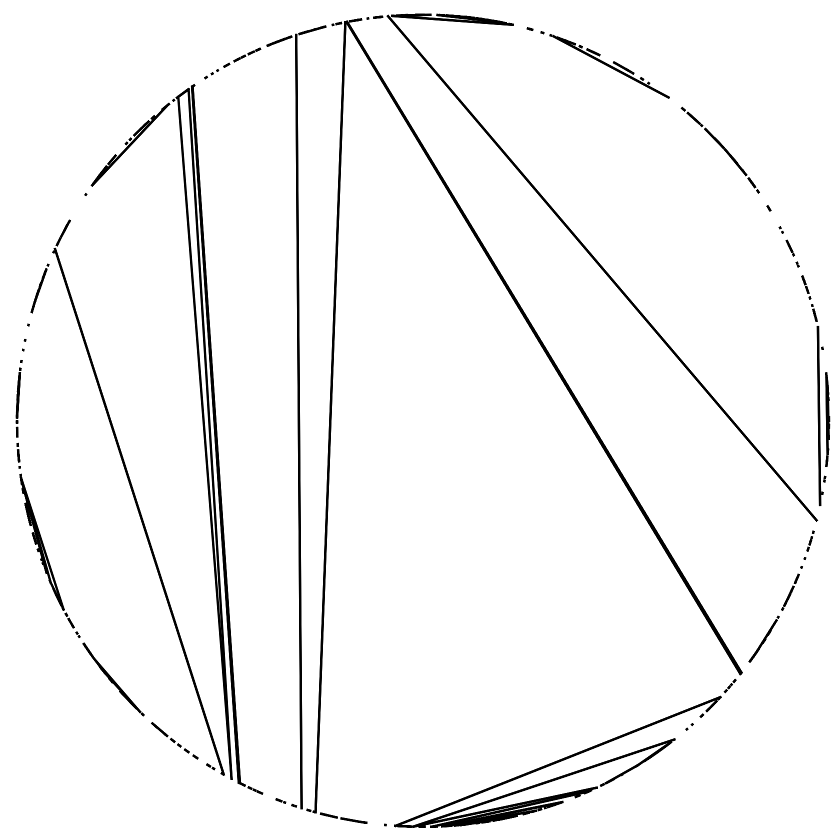
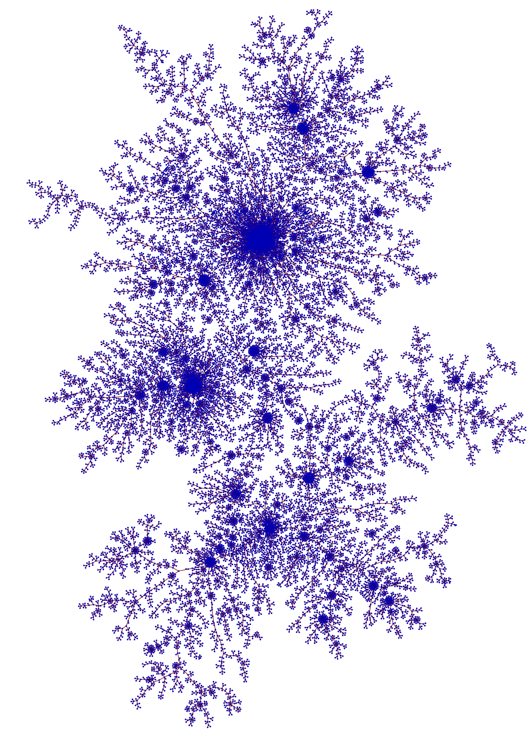
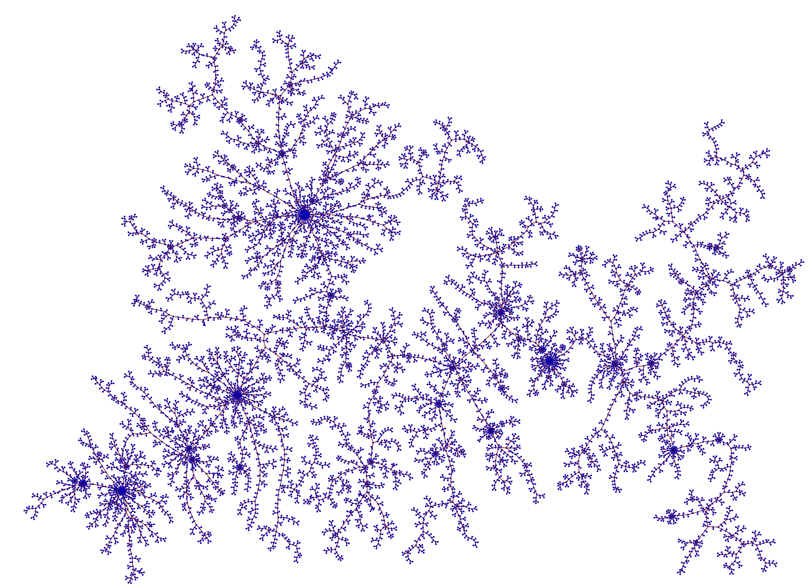
Comportement asymptotique de grandes structures discrètes aléatoires

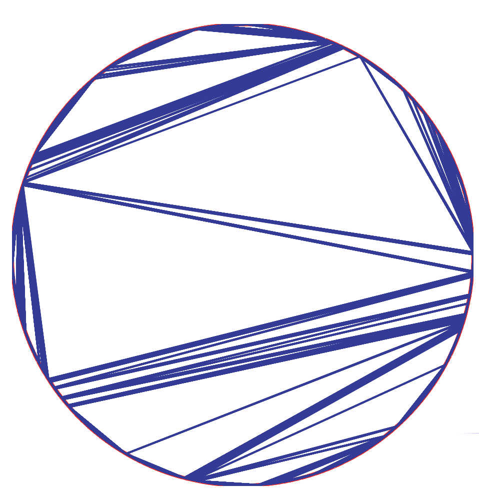












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CNRS & École polytechnique

Séminaire Flajolet – IHP – Février 2018

Questions: minimal factorizations

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Consider the set

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⇝ Question: for n large, what does a typical minimal factorization look like?

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To answer this question, a possibility is to find a continuous object X such that $X_n \to X$ as $n \to \infty$.

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⇝ *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging to X, then X_n and Y_n "roughly" have the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (Z, d) (complete separable).
- *What is the sense of this convergence when these objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left[F(X)\right]$$

for every continuous bounded function $F : Z \to \mathbb{R}$.

Outline

I. Trees

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⇝ Probability: trees are elementary pieces of various models of random graphs, having rich probabilistic properties.

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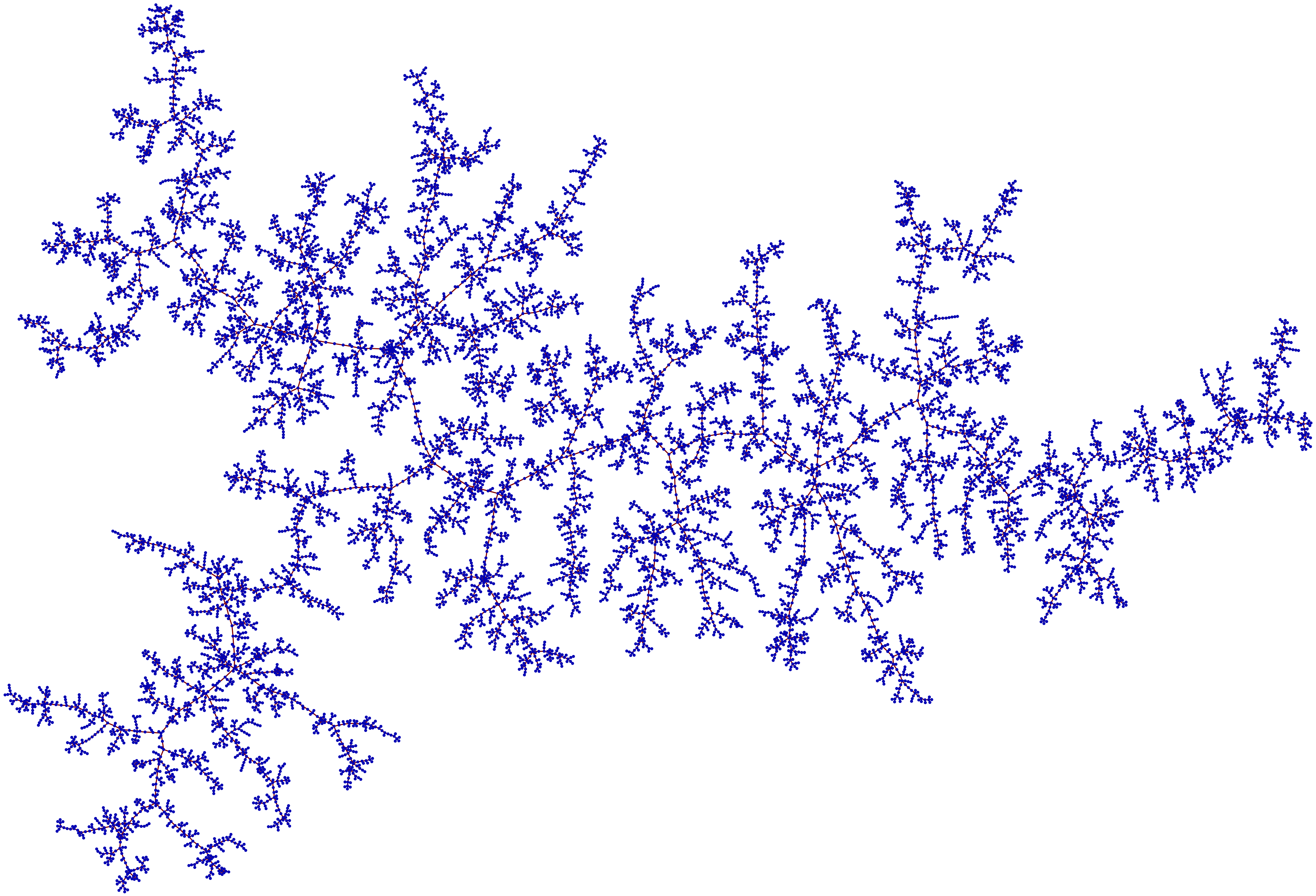

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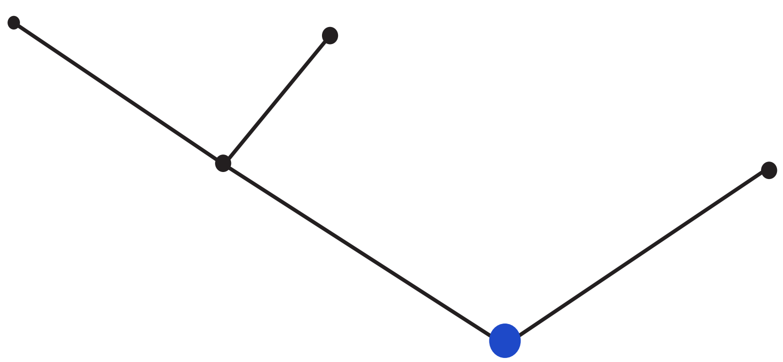

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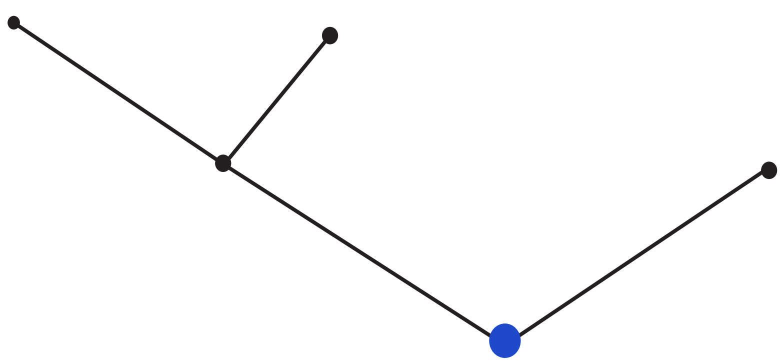

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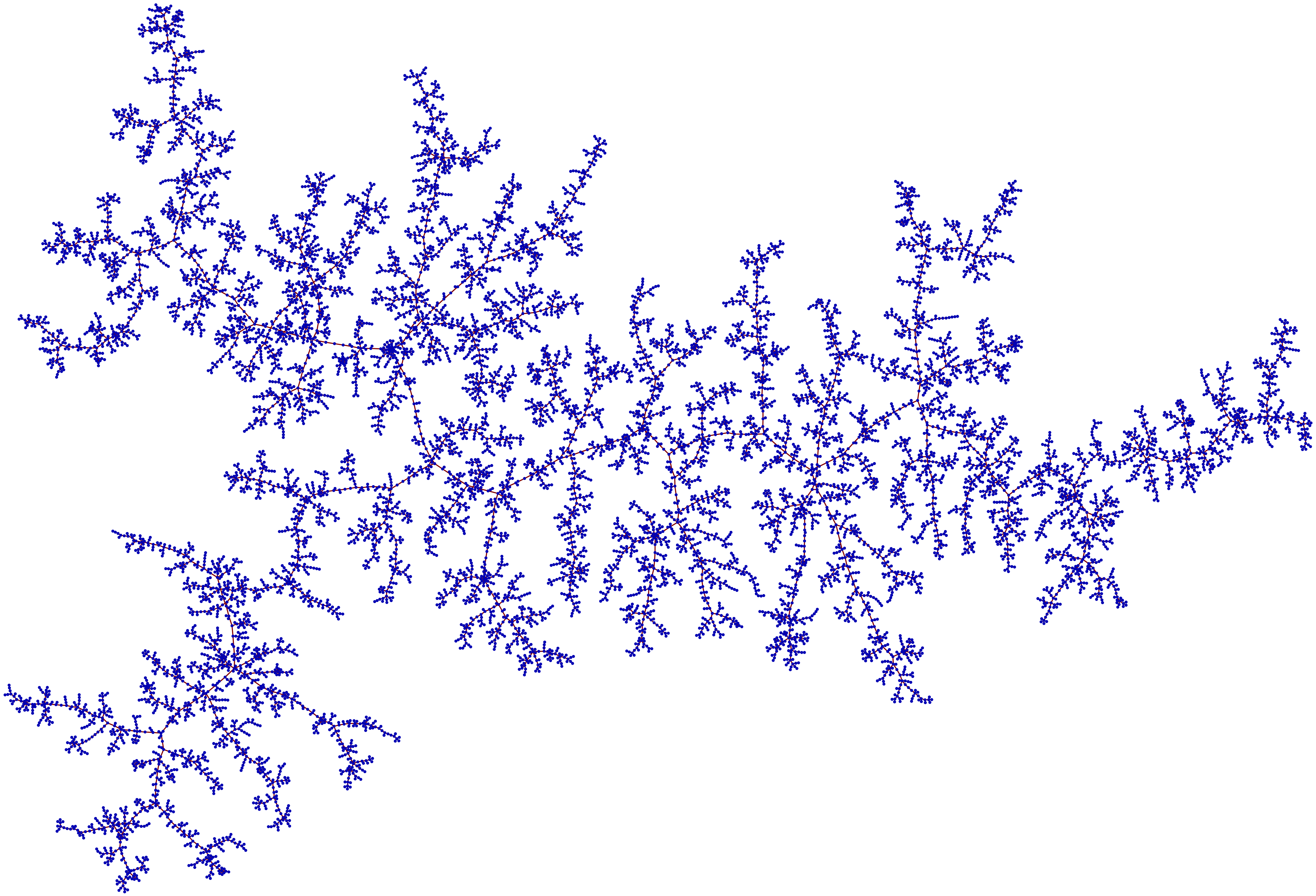
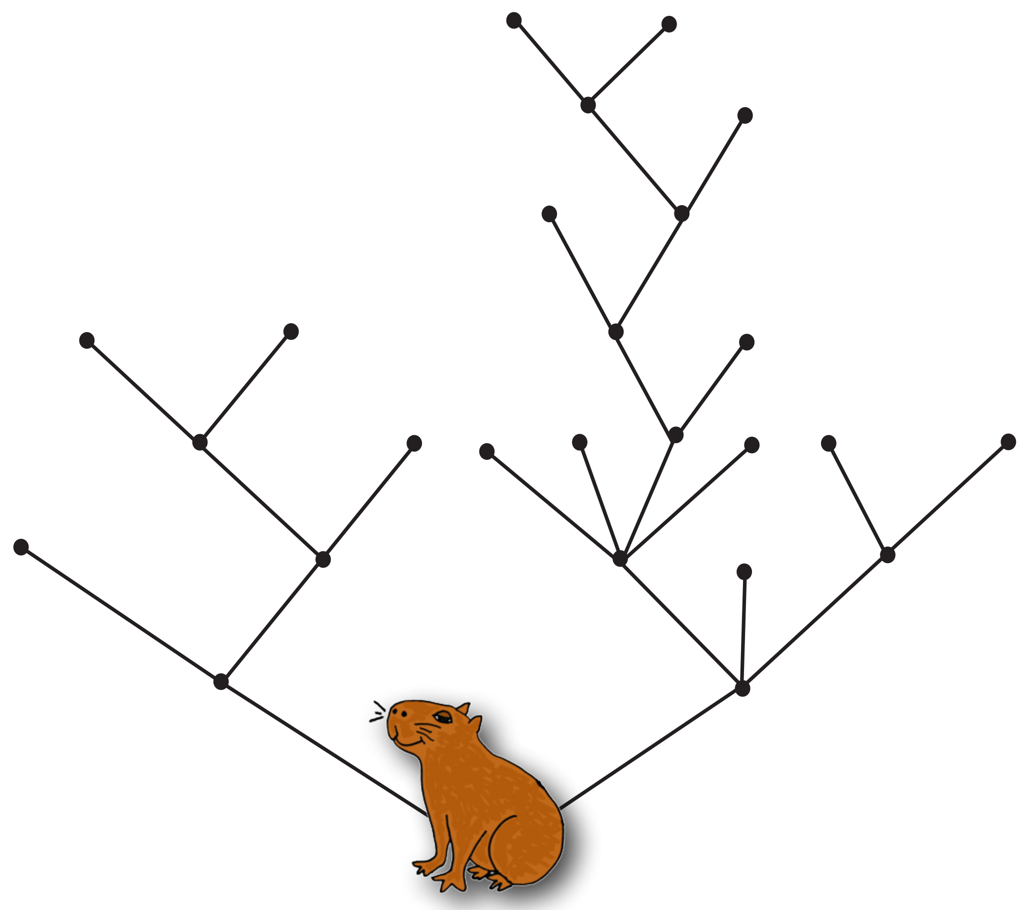
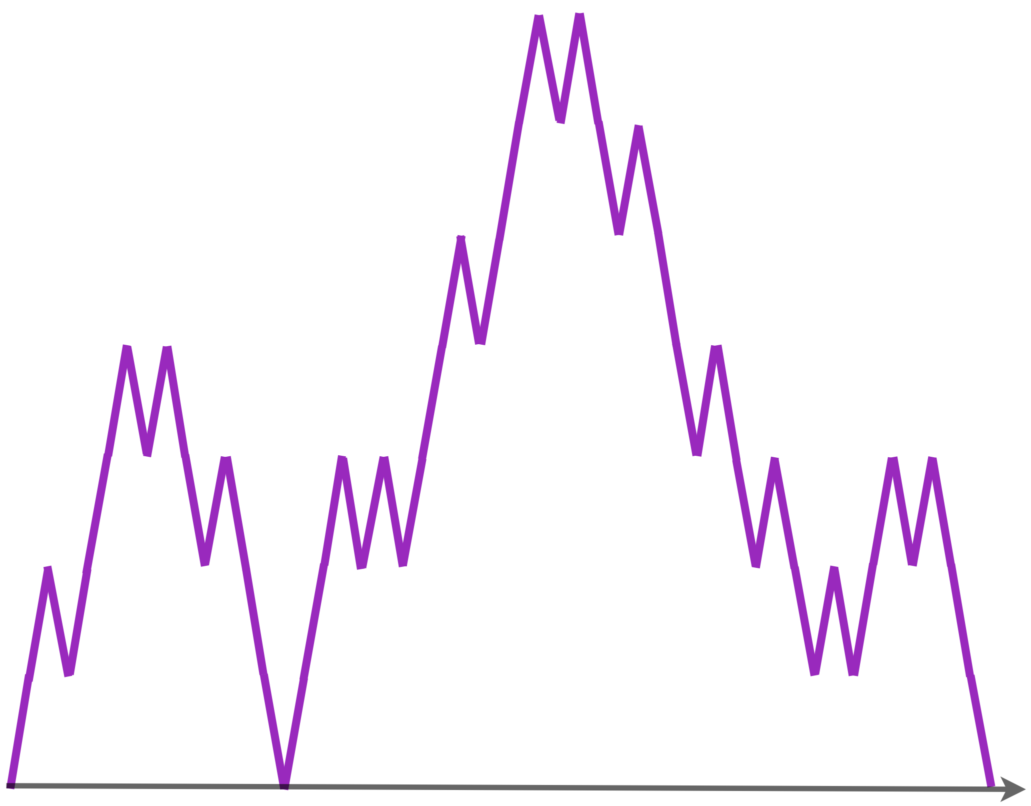
⇝ If $\mu(i) = \frac{1}{2^{i+1}}$ for $i \geqslant 0$, a BGW tree conditioned on having n vertices follows the uniform distribution on the set of all plane trees with n vertices!



Coding a tree by its contour function

Code a tree τ by its contour function $C(\tau)$:





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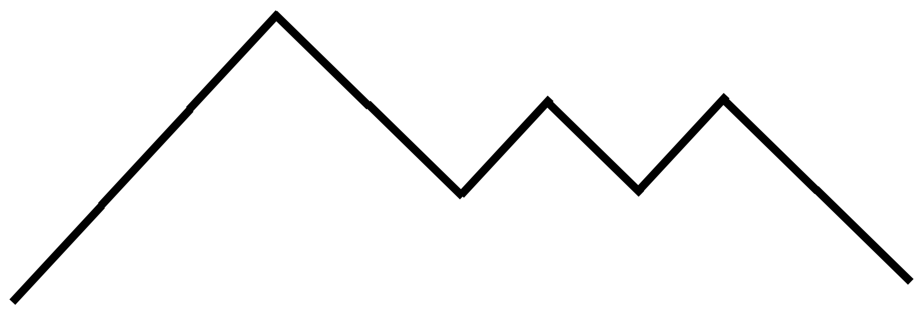
Knowing the contour function, it is easy to reconstruct the tree:



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Theorem (Aldous '93)

We have:

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(t_n)\right)_{0 \leqslant t \leqslant 1} \xrightarrow[n\to\infty]{(d)} \left(\frac{2}{\sigma} \cdot \mathbb{e}(t)\right)_{0 \leqslant t \leqslant 1},$$

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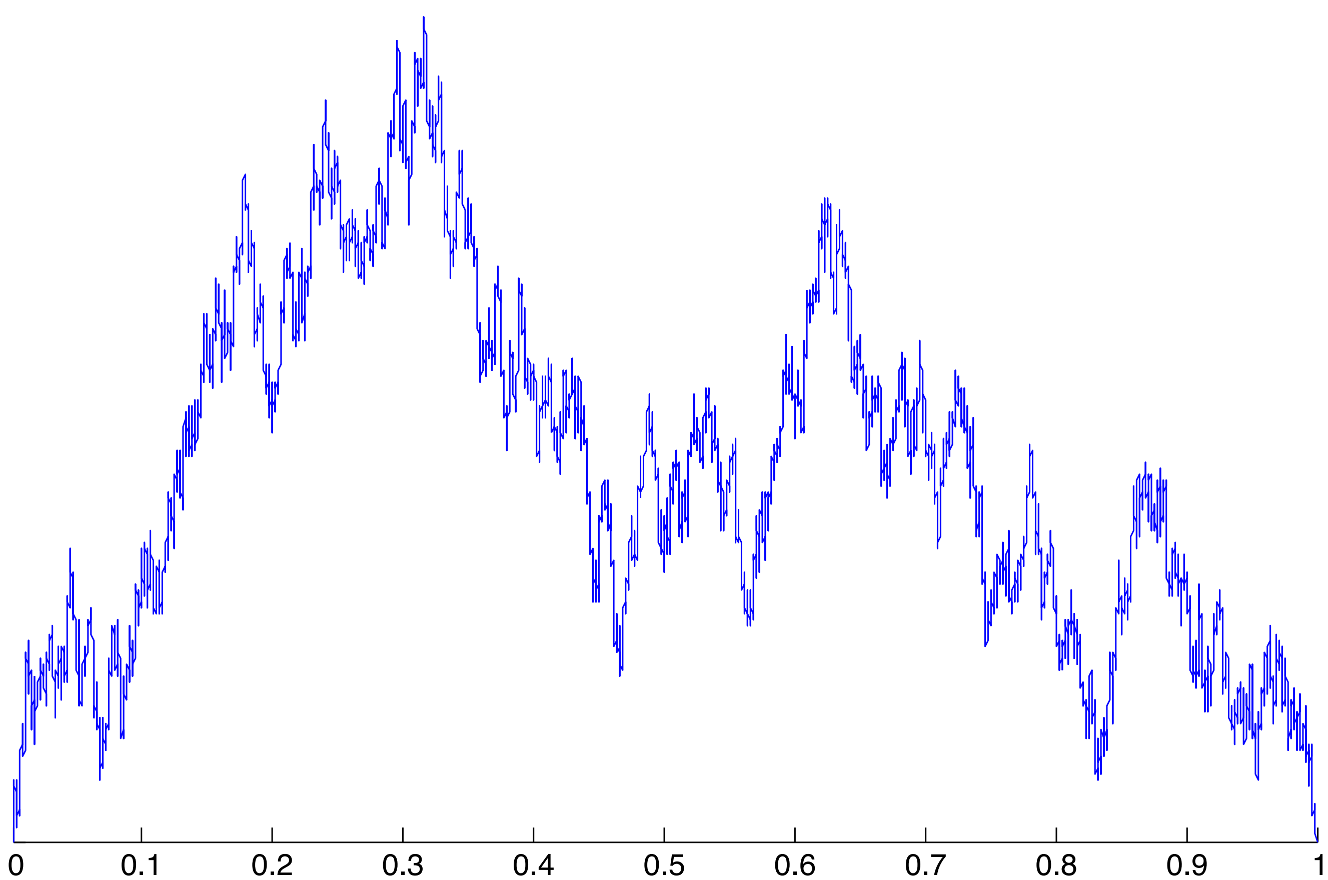

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⇝ Ideas: code t_n by another function (Lukasiewicz path), which is a (conditioned) random walk, use (a conditioned) Donsker's invariance principle, go back to the contour function (Duquesne & Le Gall, Marckert & Mokkadem).

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where $\mathbb{e}$ *is the normalized Brownian excursion.*

⇝ Consequence 1: for every $a > 0$,

$$\mathbb{P}\left[\frac{\sigma}{2}\cdot \textbf{Height}(t_n) > a\cdot\sqrt{n}\right] \xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty}(4k^2a^2-1)e^{-2k^2a^2}.$$

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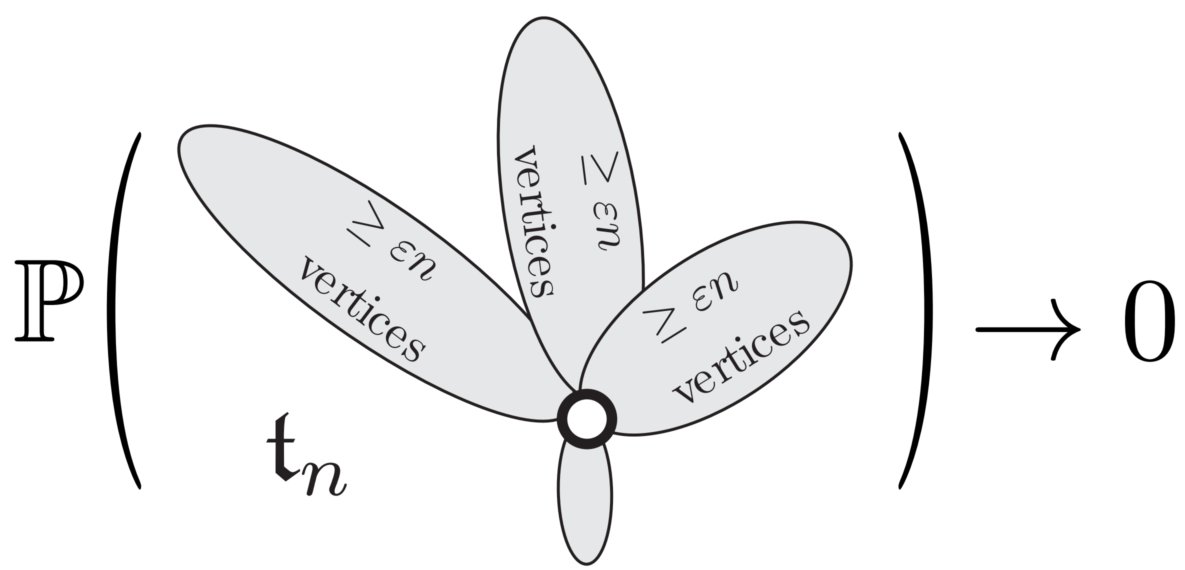
⇝ Consequence 2: for every $\varepsilon > 0$,

$$\mathbb{P}\left(\text{there exists a vertex of } t_n \text{ with 3 grafted subtrees of sizes } \geqslant \varepsilon n\right) \to 0.$$



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- different families of tree-like structures: stack triangulations (Albenque & Marckert), graphs from subcritical classes (Panagiotou, Stufler & Weller), dissections (Curien, Haas & K), various maps (Janson & Stefánsson, Bettinelli, Caraceni, K & Richier).

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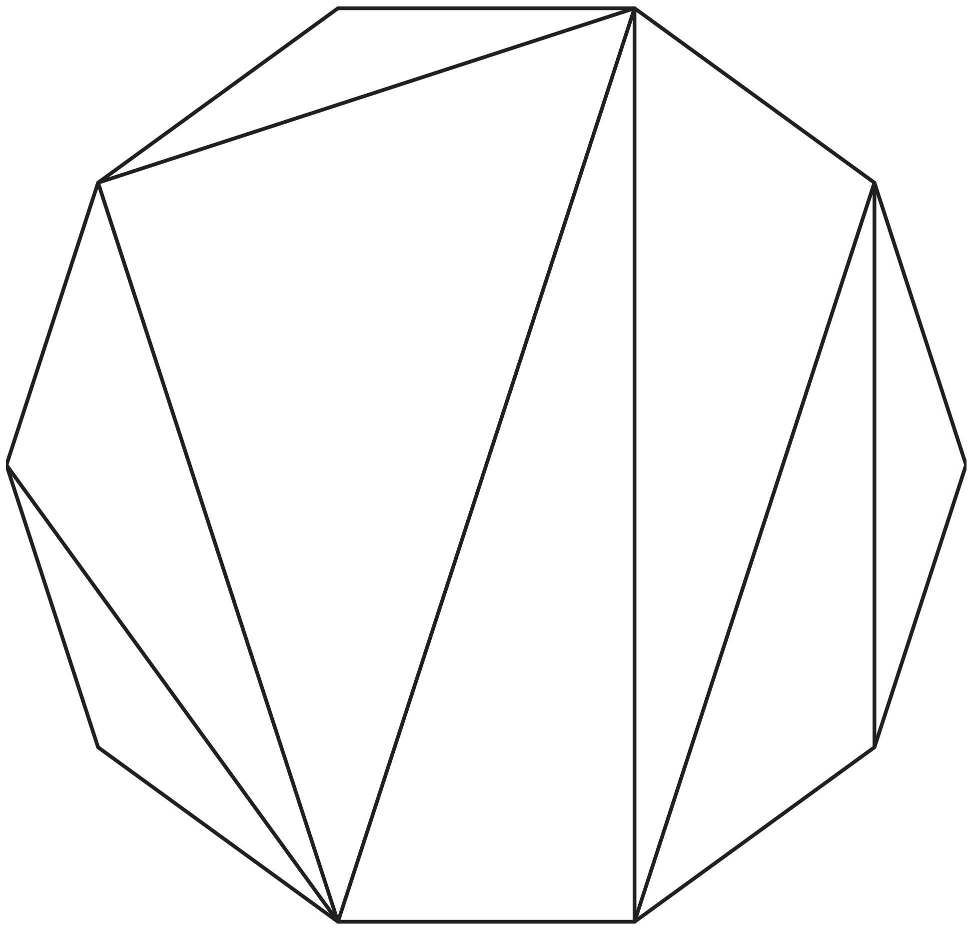

Figure: A triangulation of $\mathcal{X}_{10}$.

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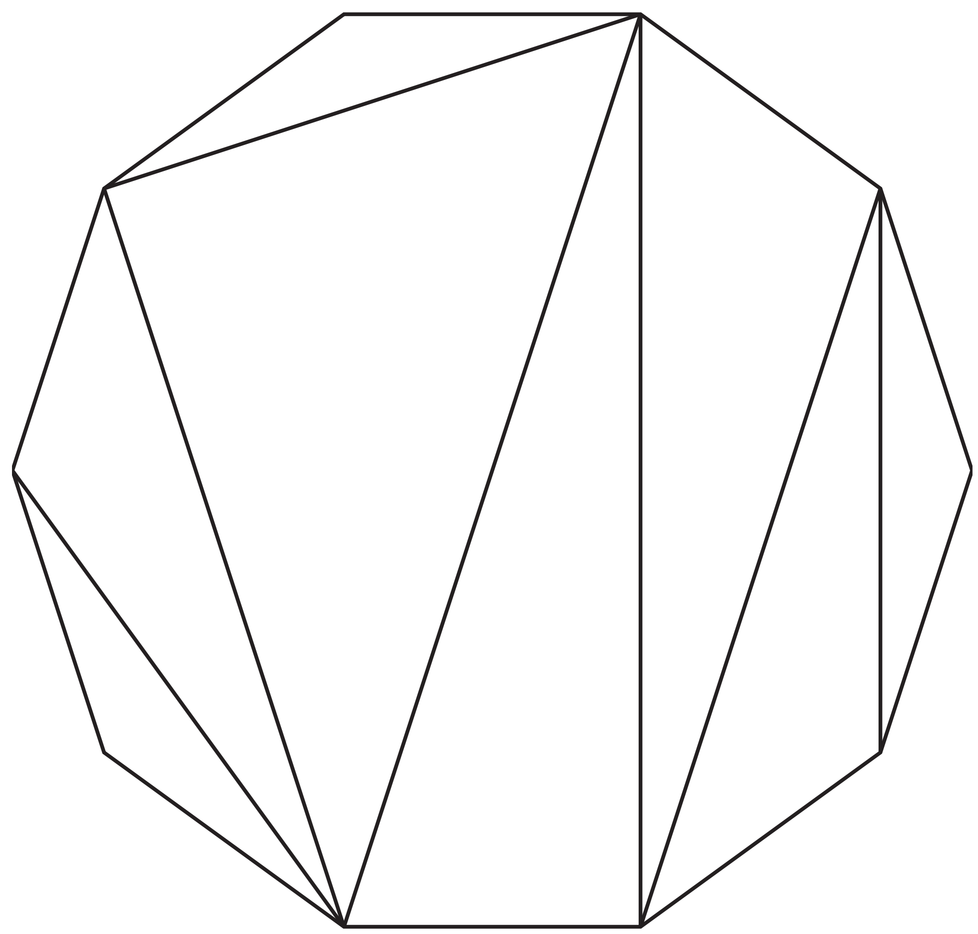

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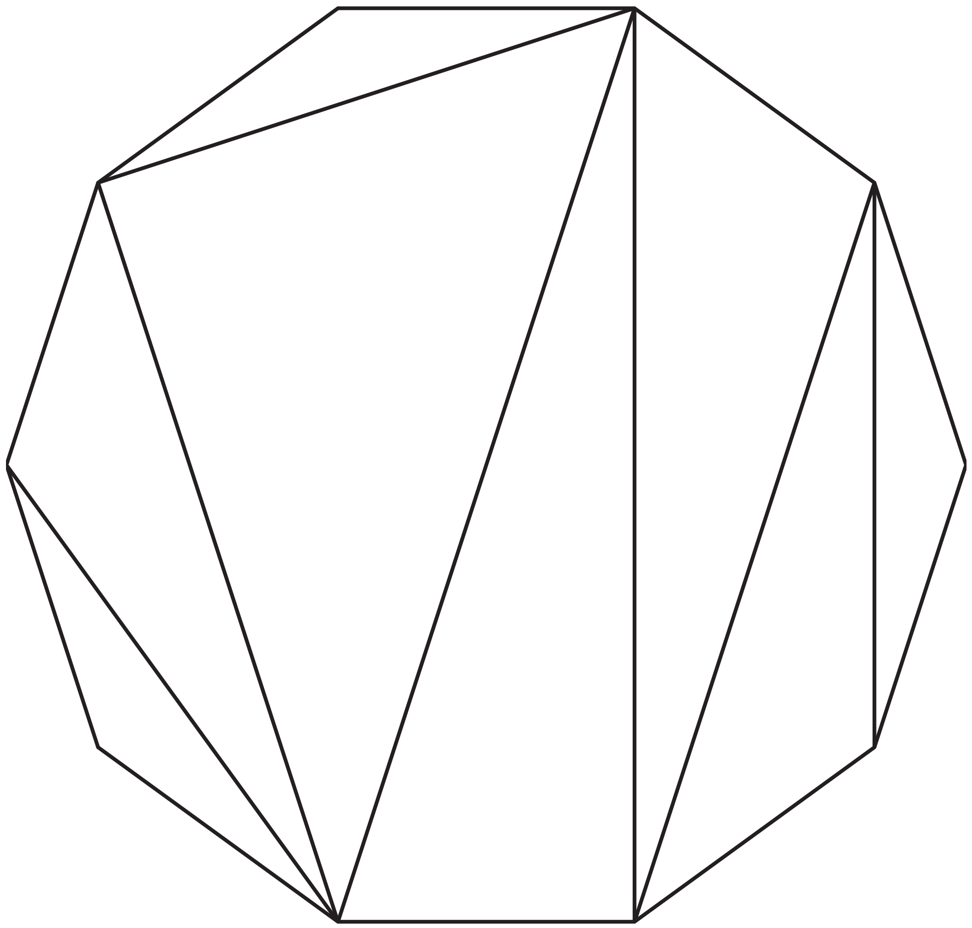

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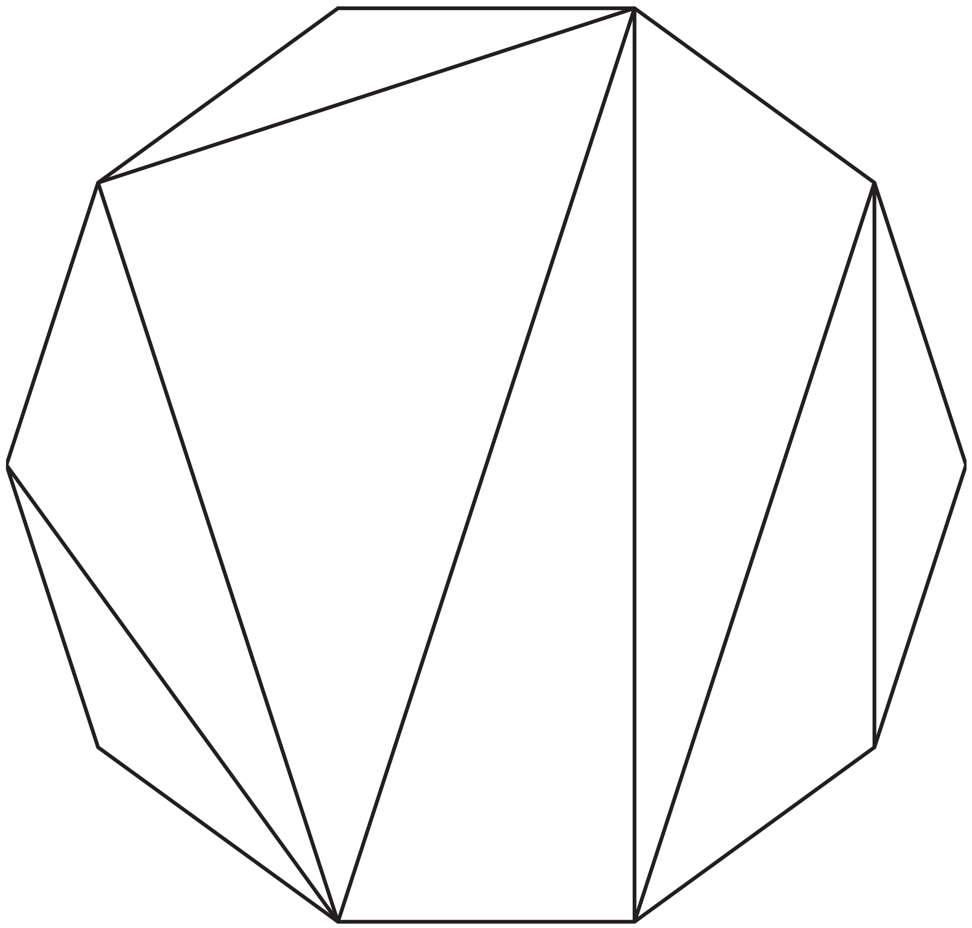

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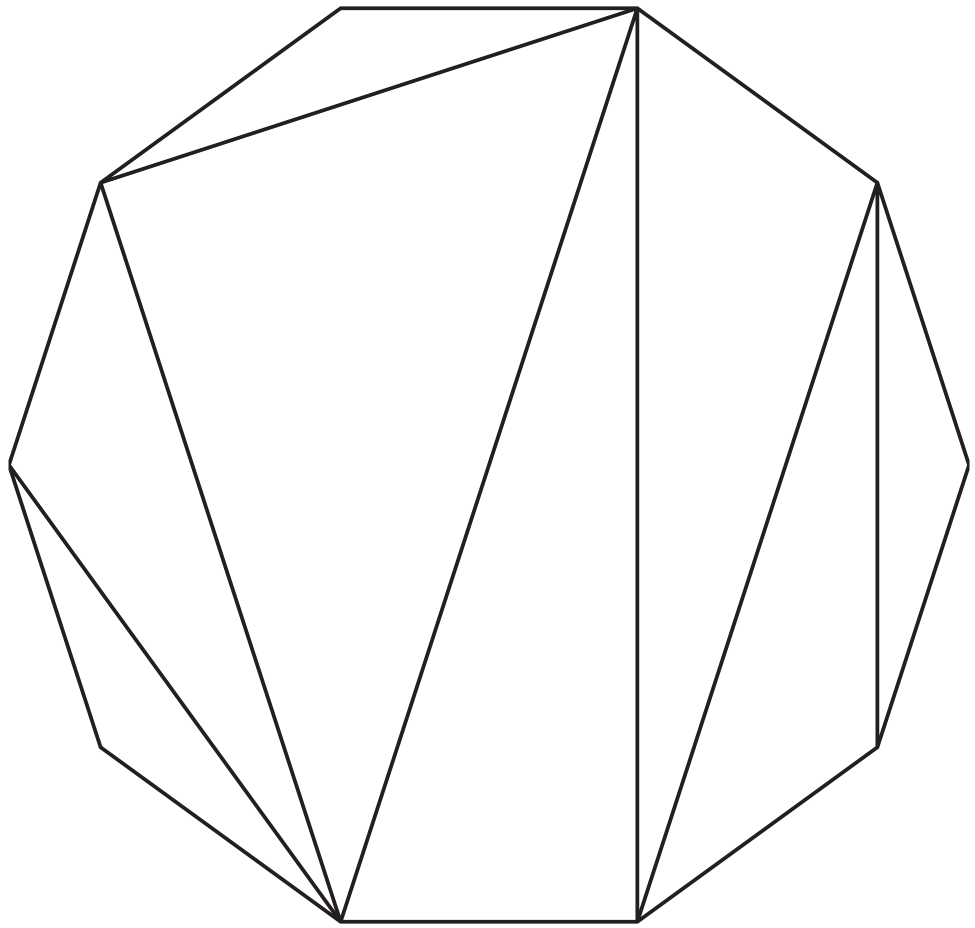
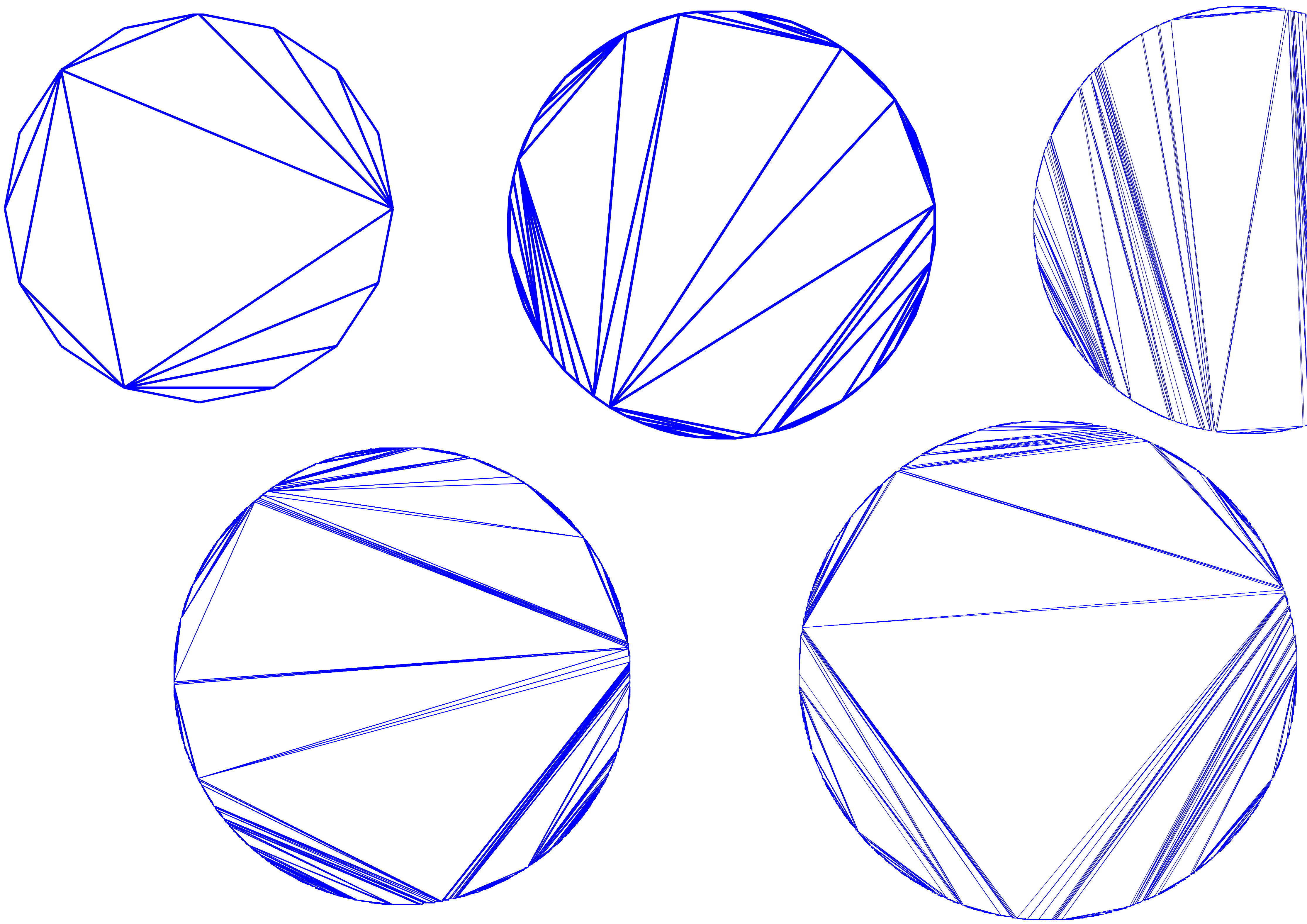


Figure: A triangulation of $\mathcal{X}_{10}$.

⇝ Question: $\#\mathcal{X}_n = \frac{1}{n-2}\binom{2n-4}{n-3}$.

⇝ Question: What does a large typical triangulation look like?

Typical triangulations



What space for triangulations?

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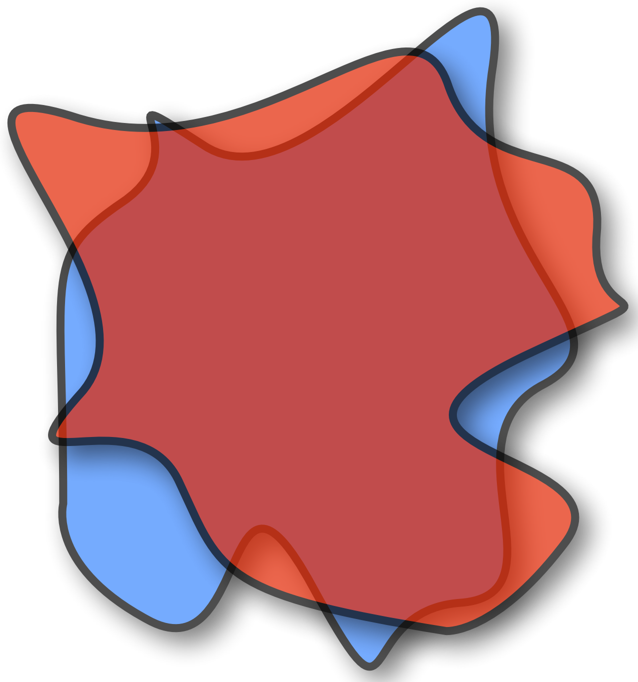

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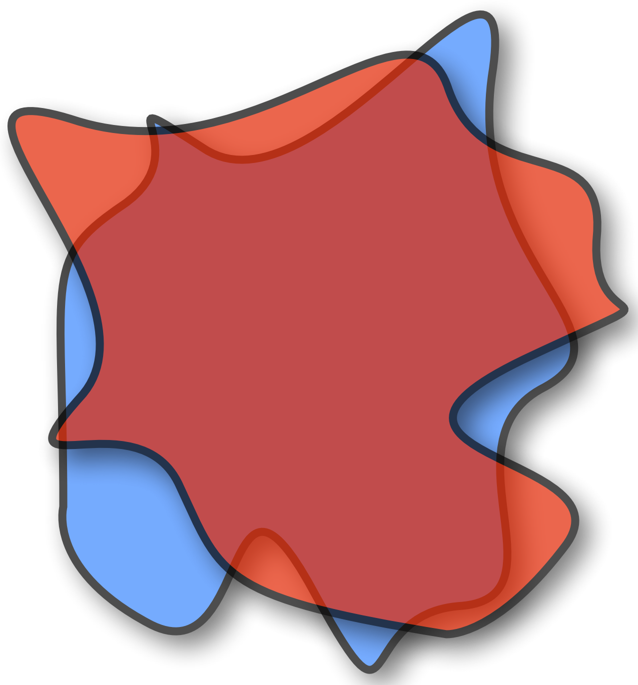

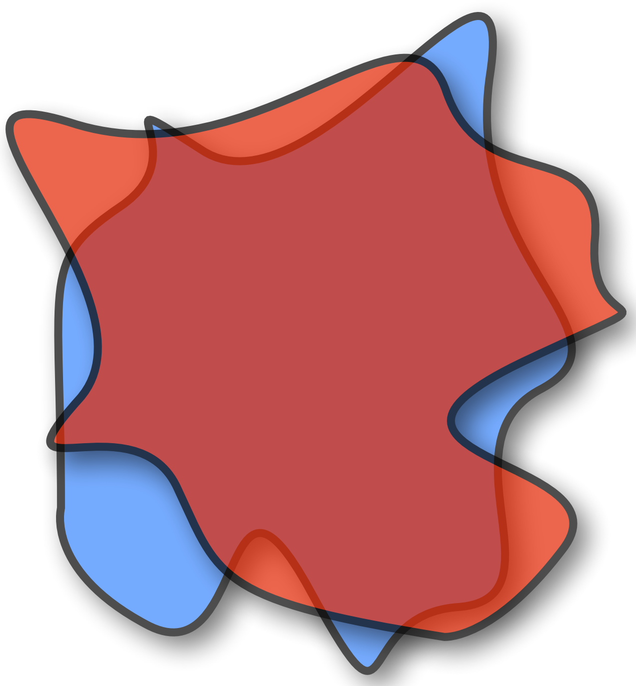

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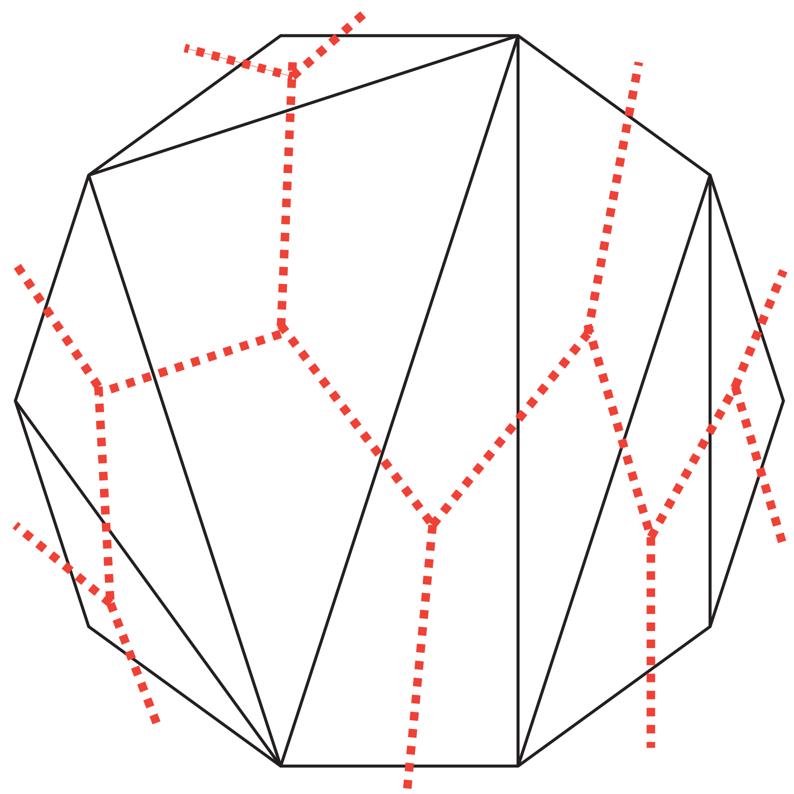

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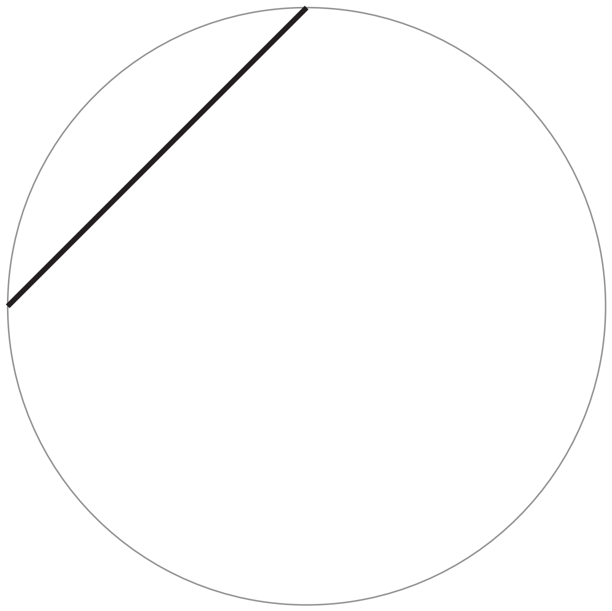

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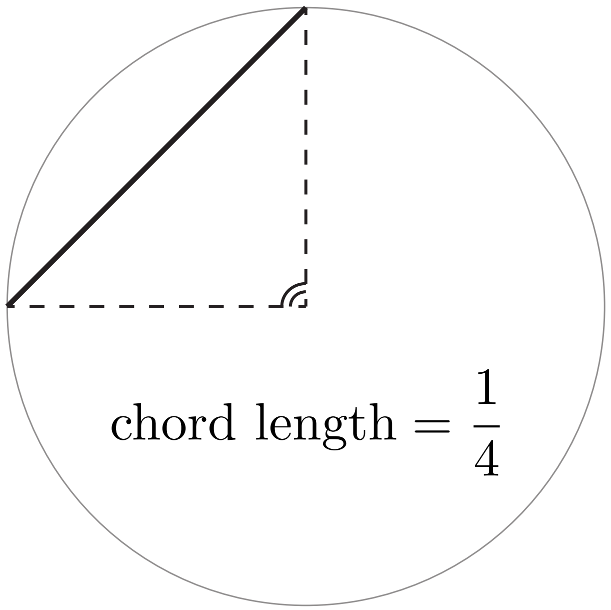

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⇝ Application (Curien & K.): study of the length of the longest chord of a uniform dissection (faces of any degree allowed).

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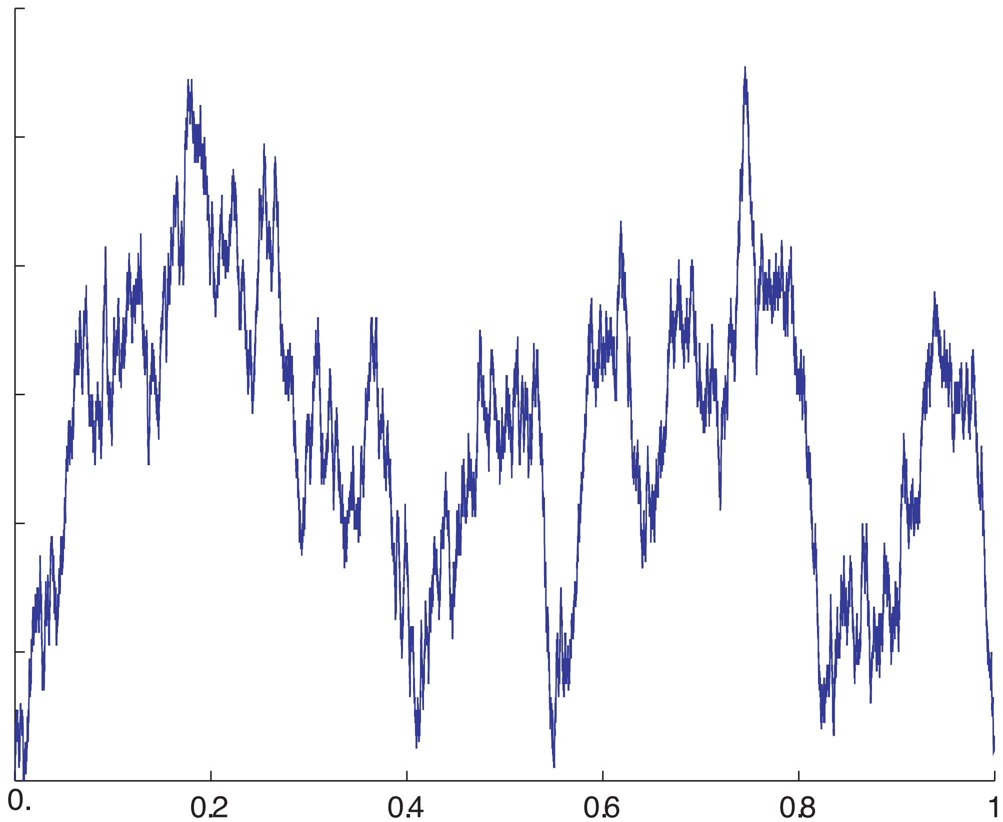

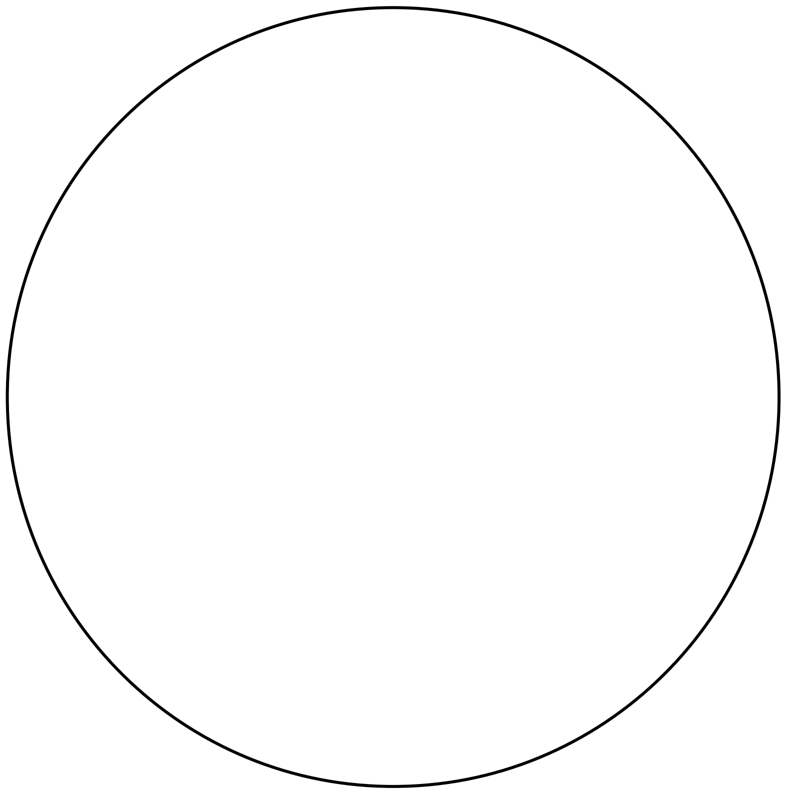

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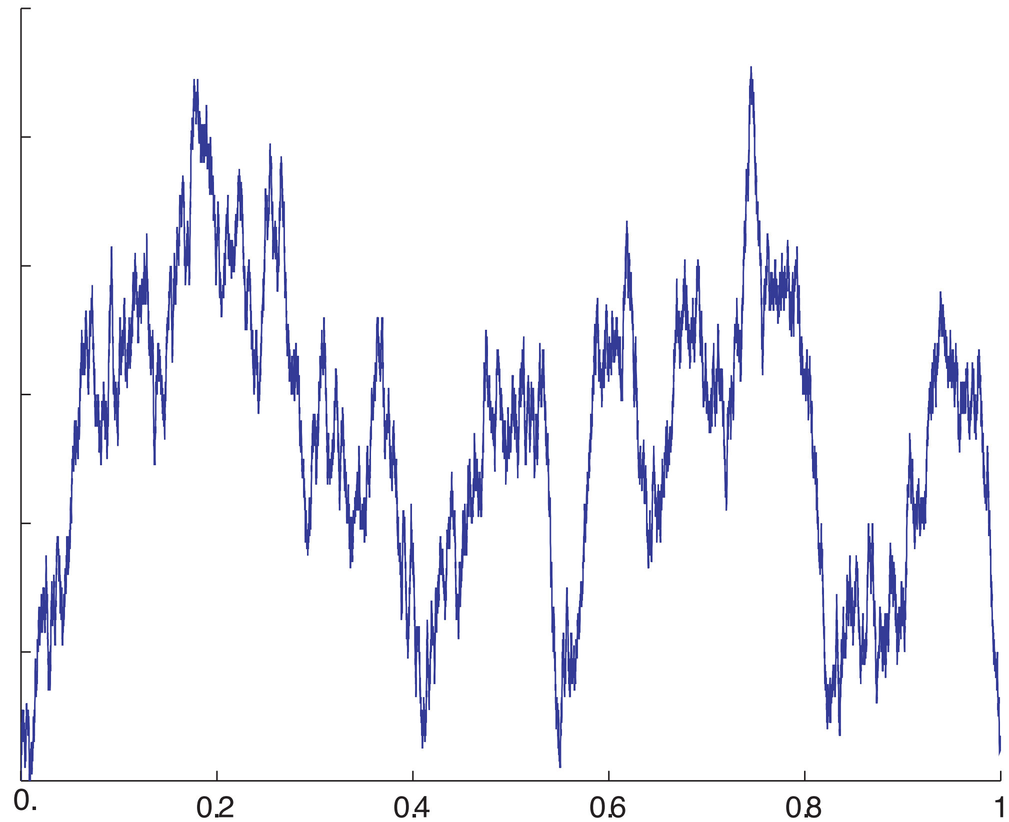

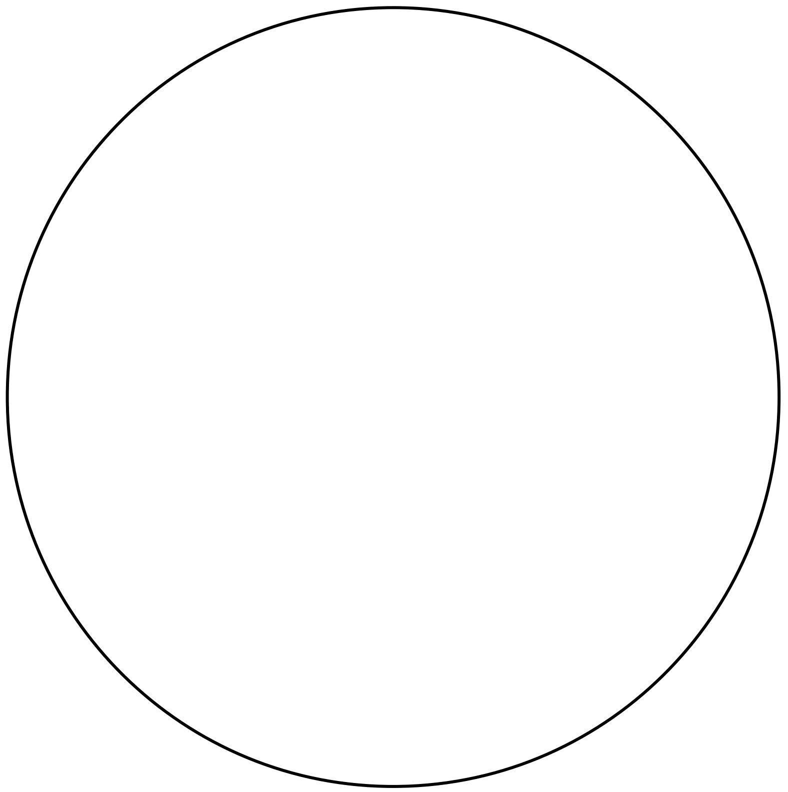

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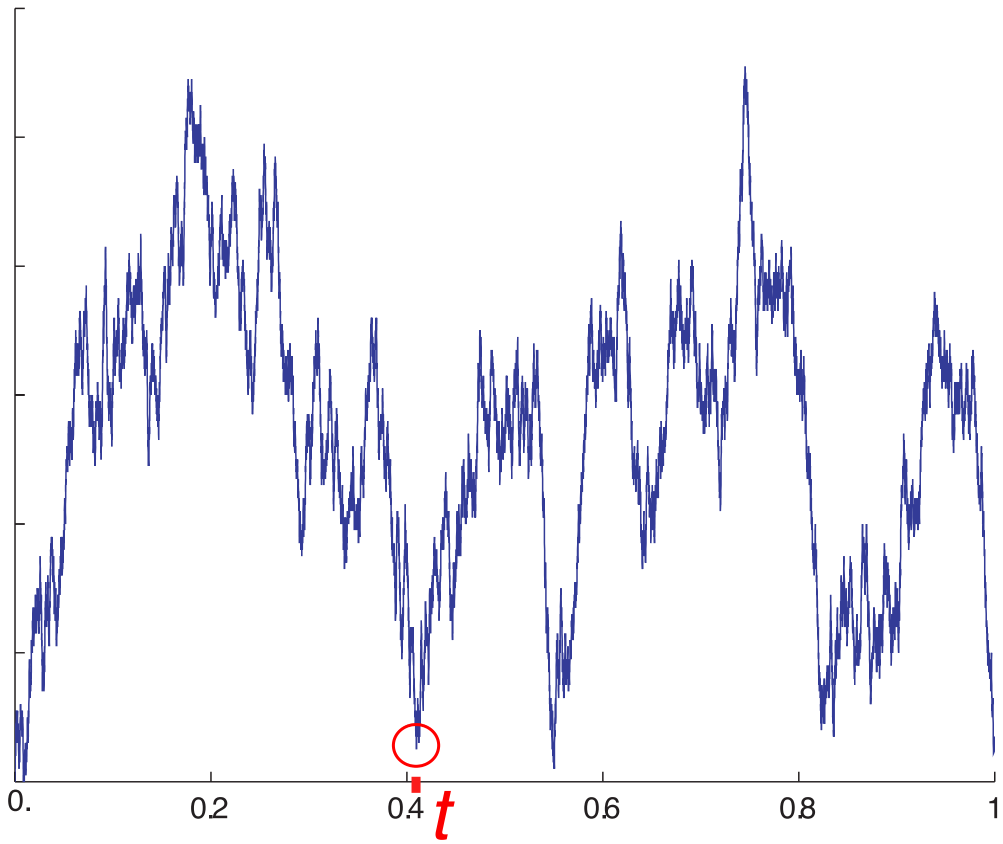

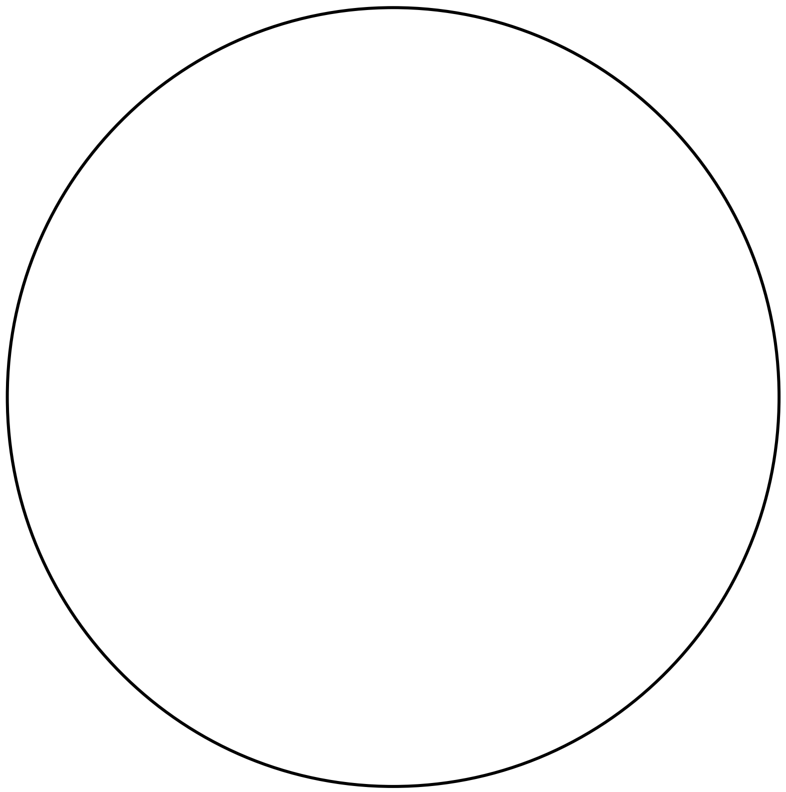

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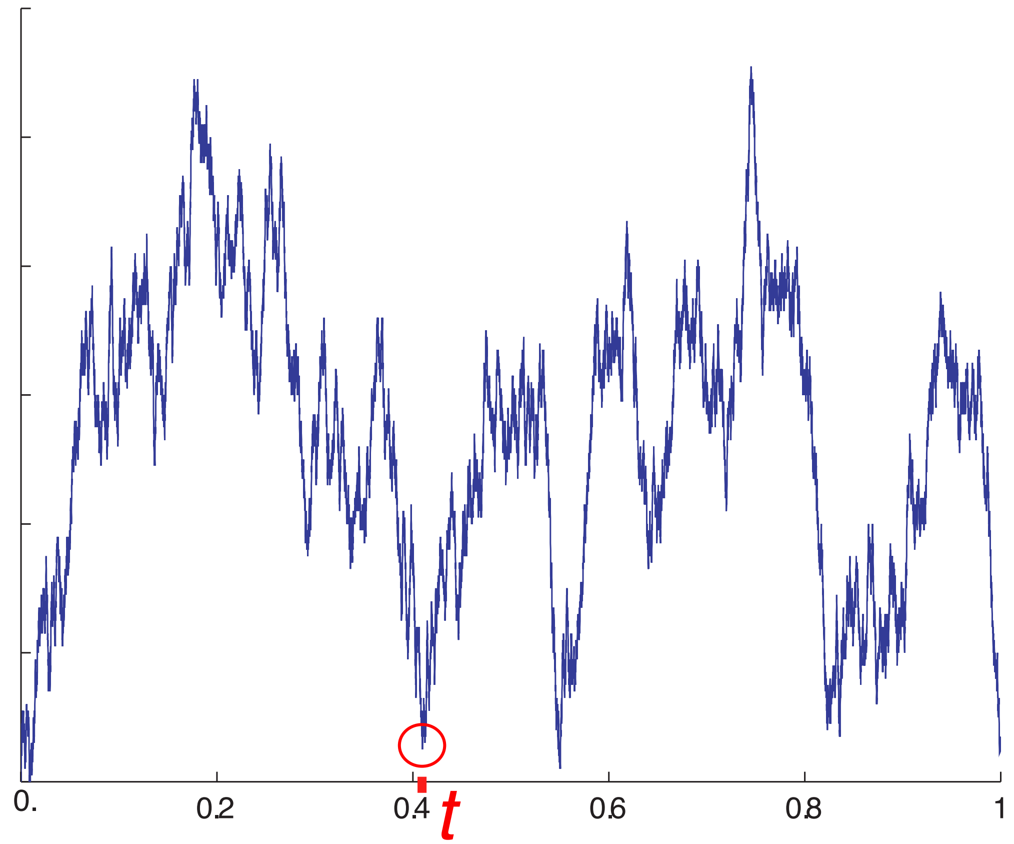

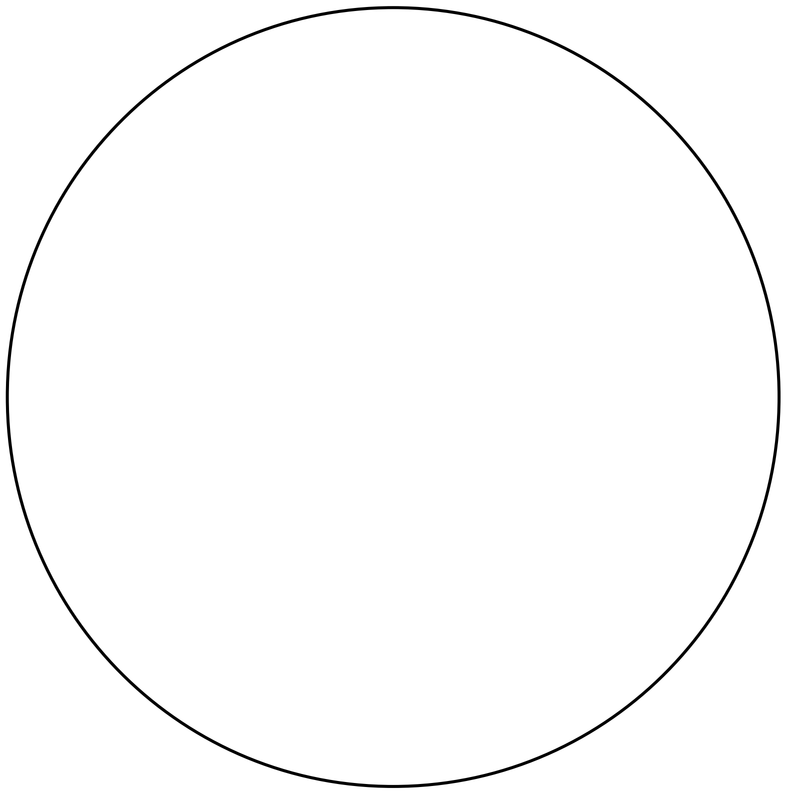

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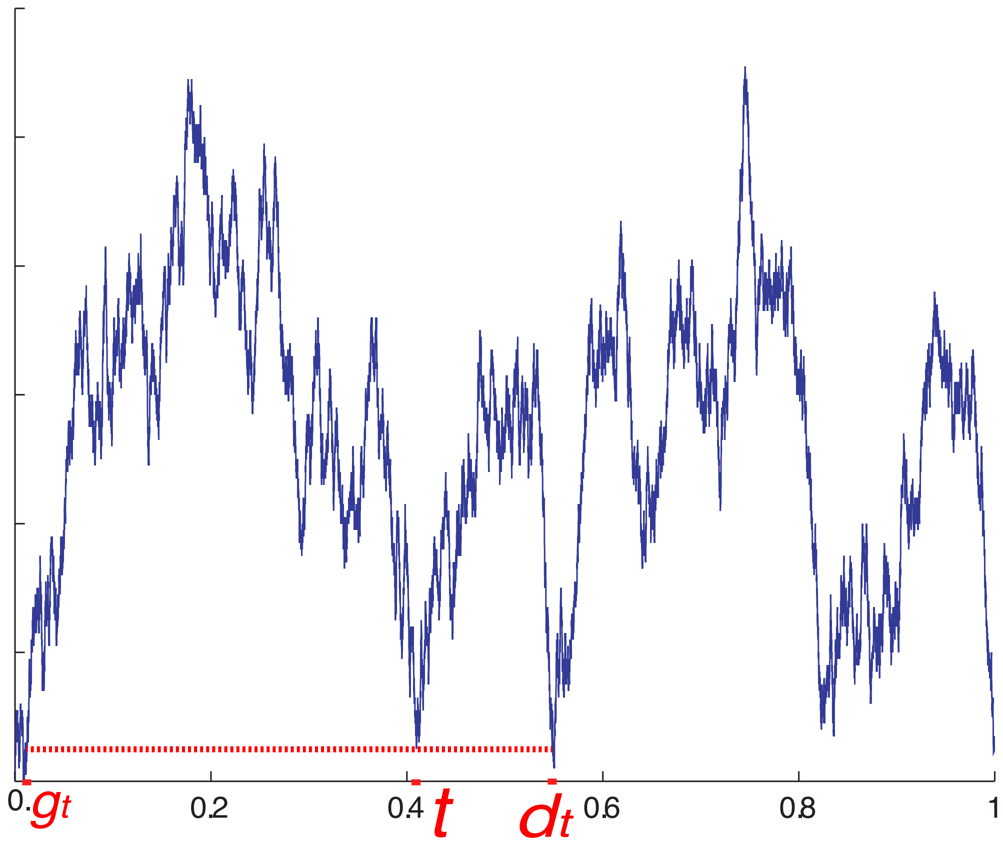

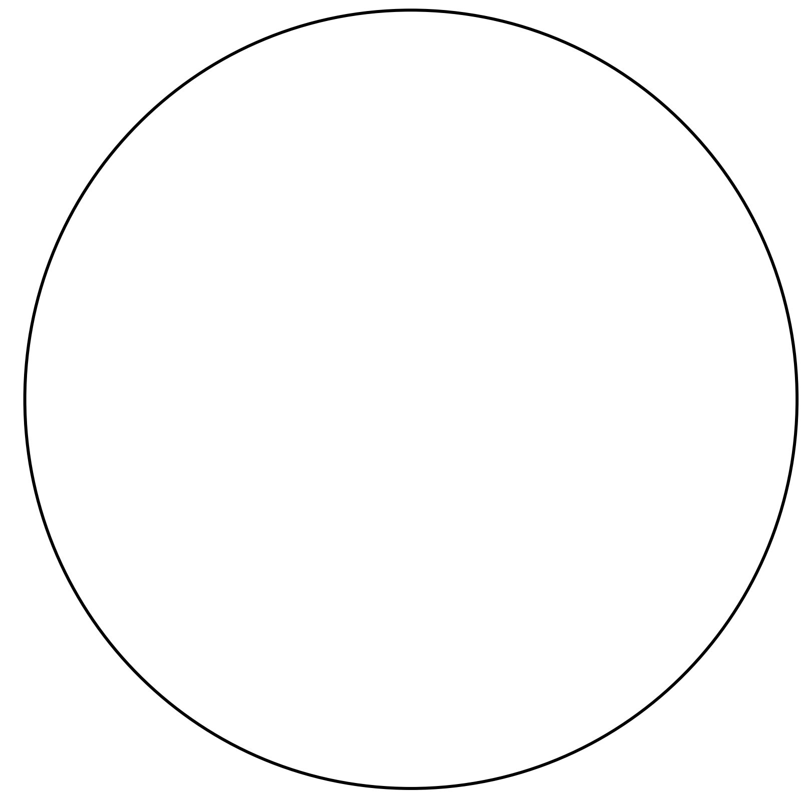

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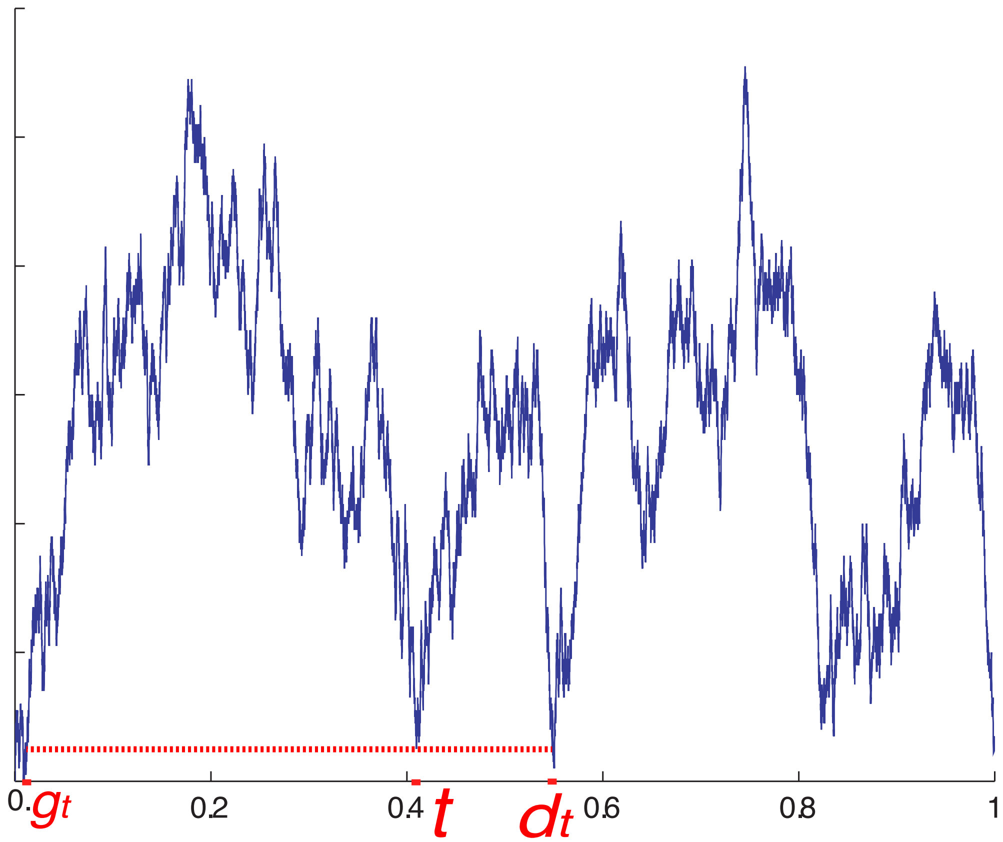

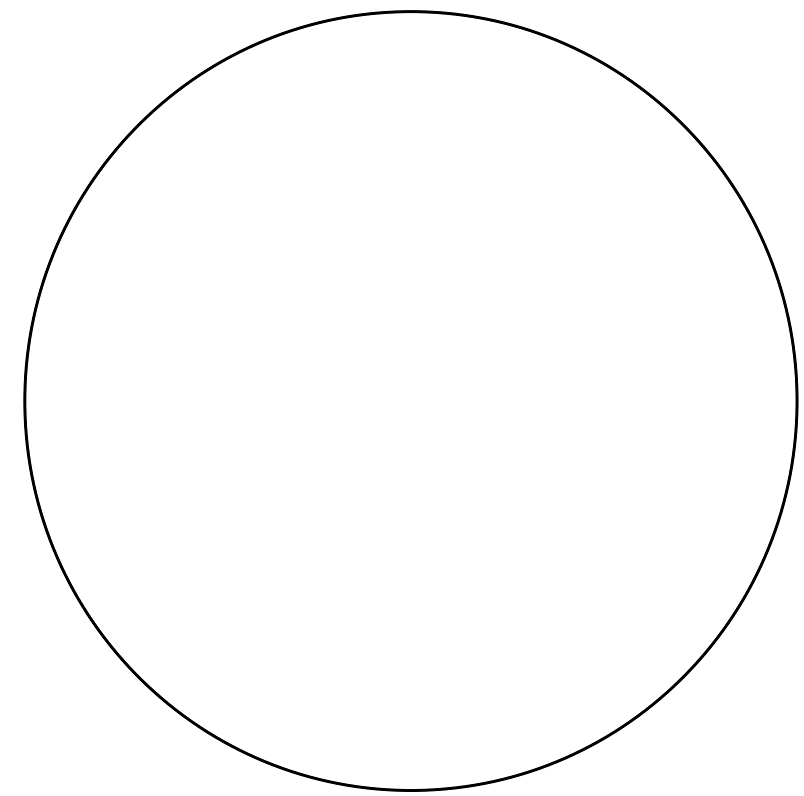

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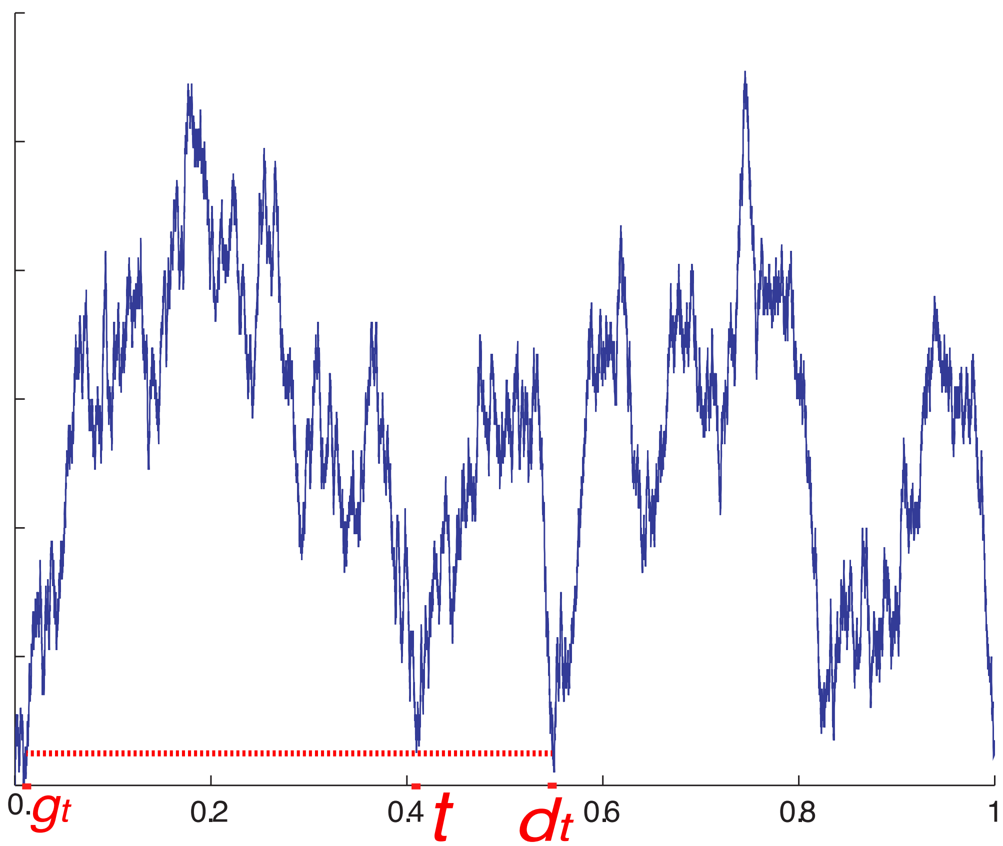

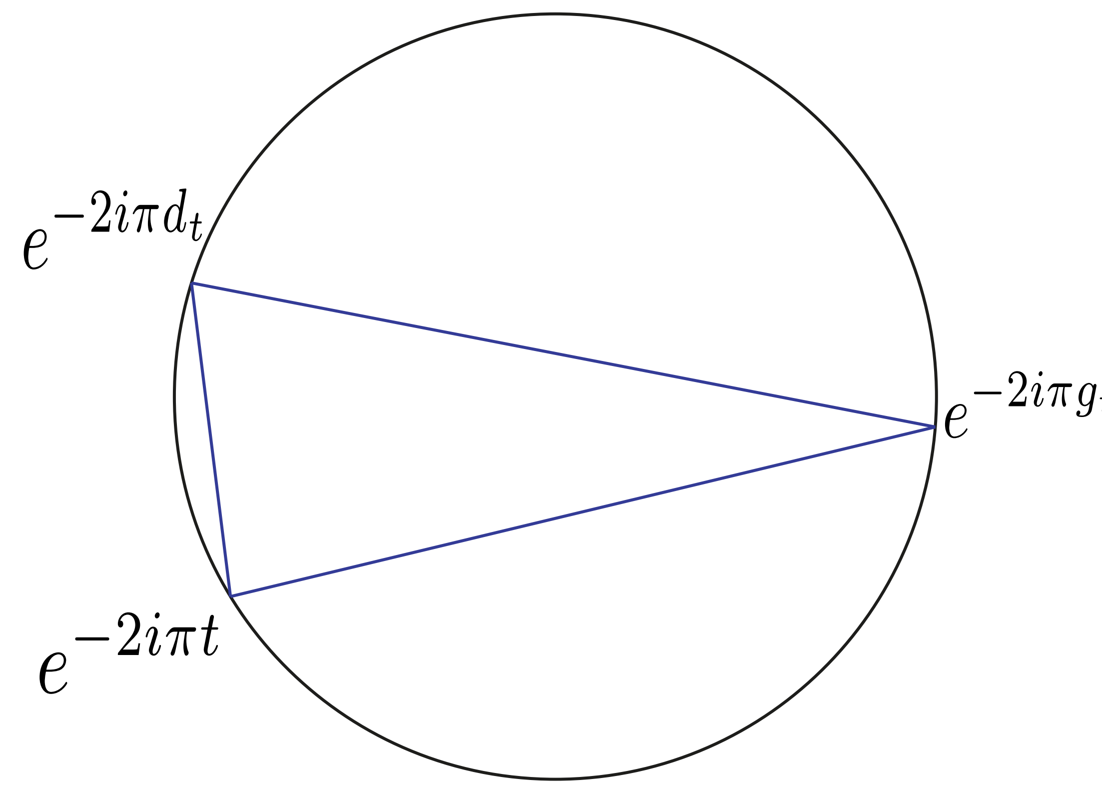

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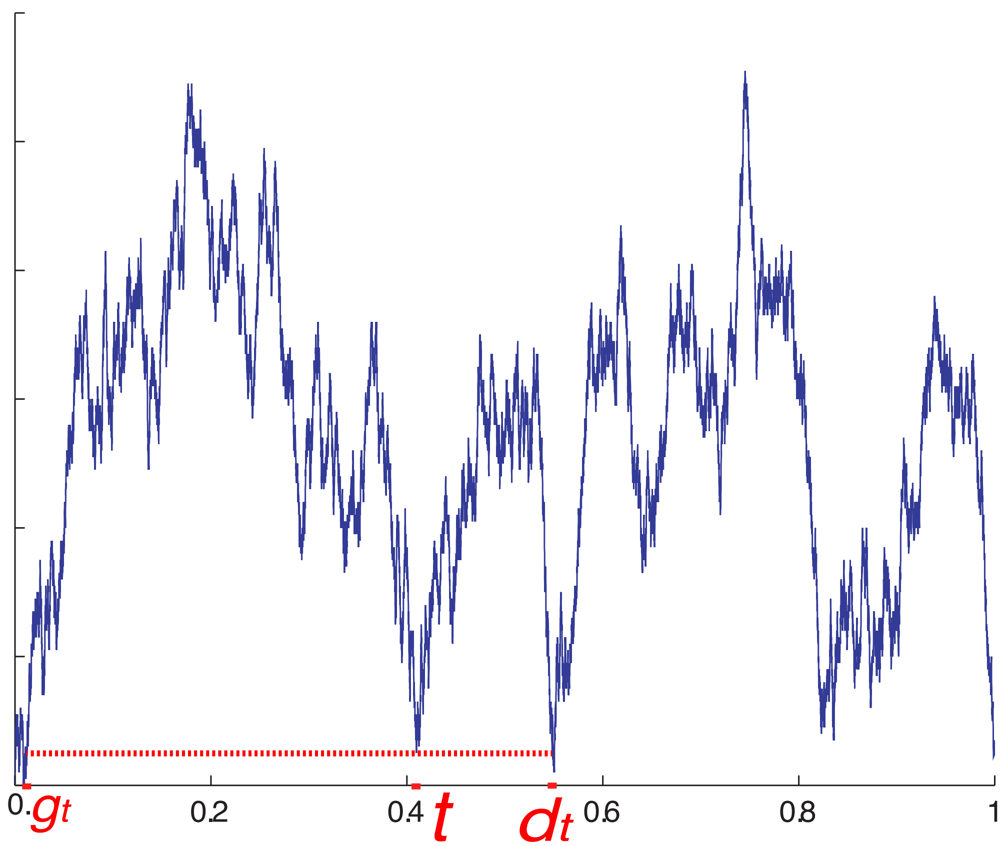

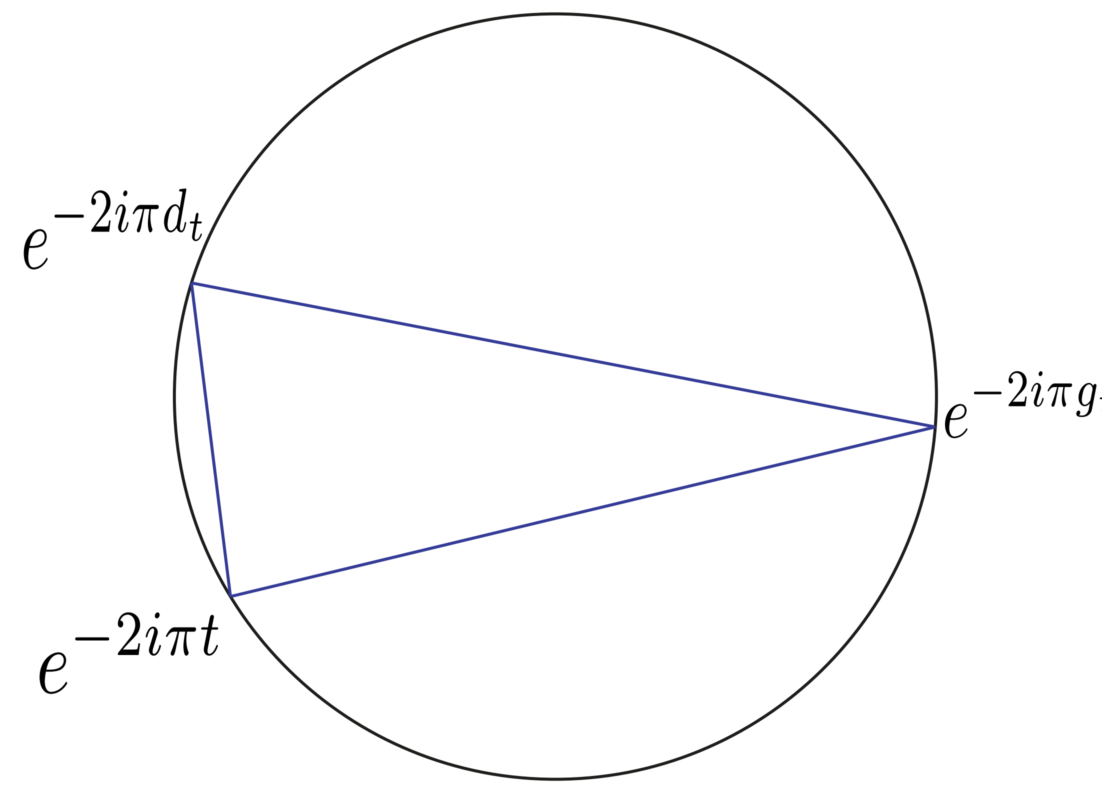

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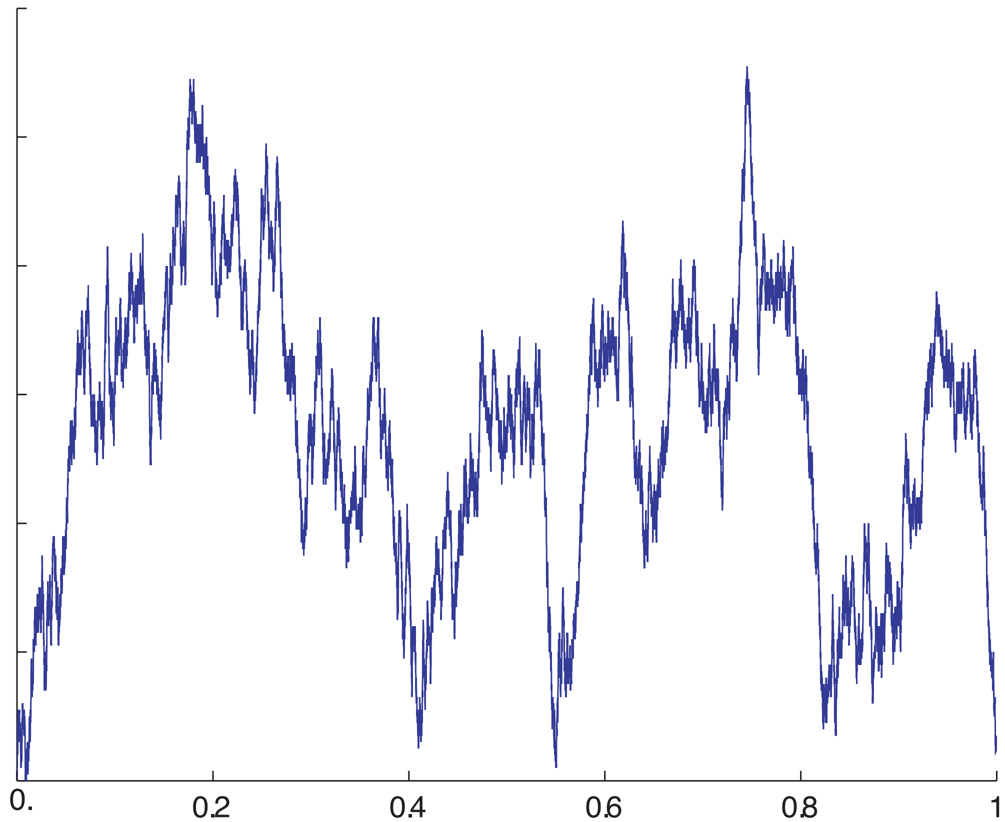

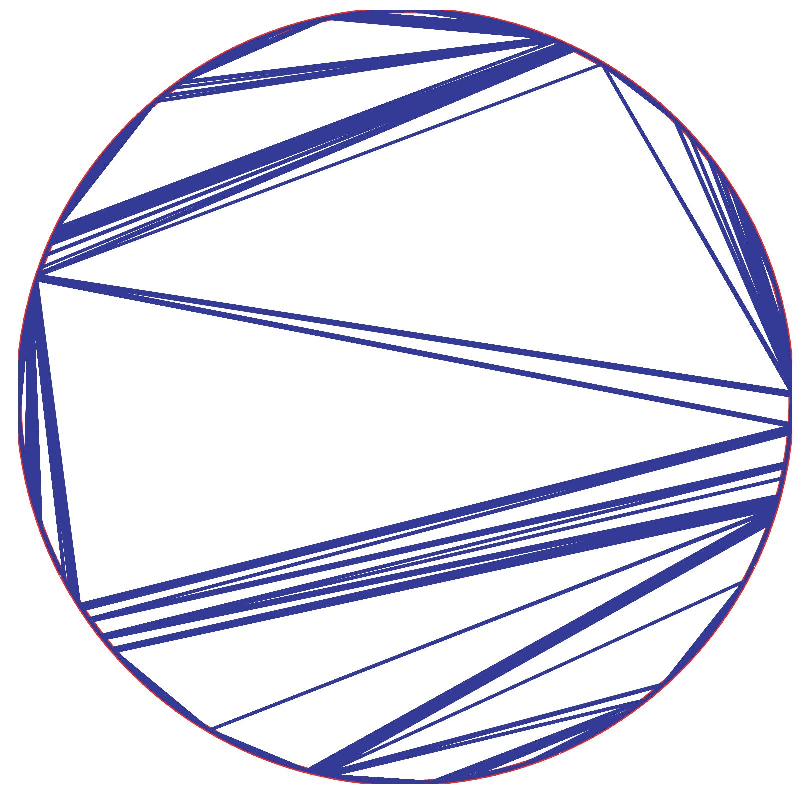

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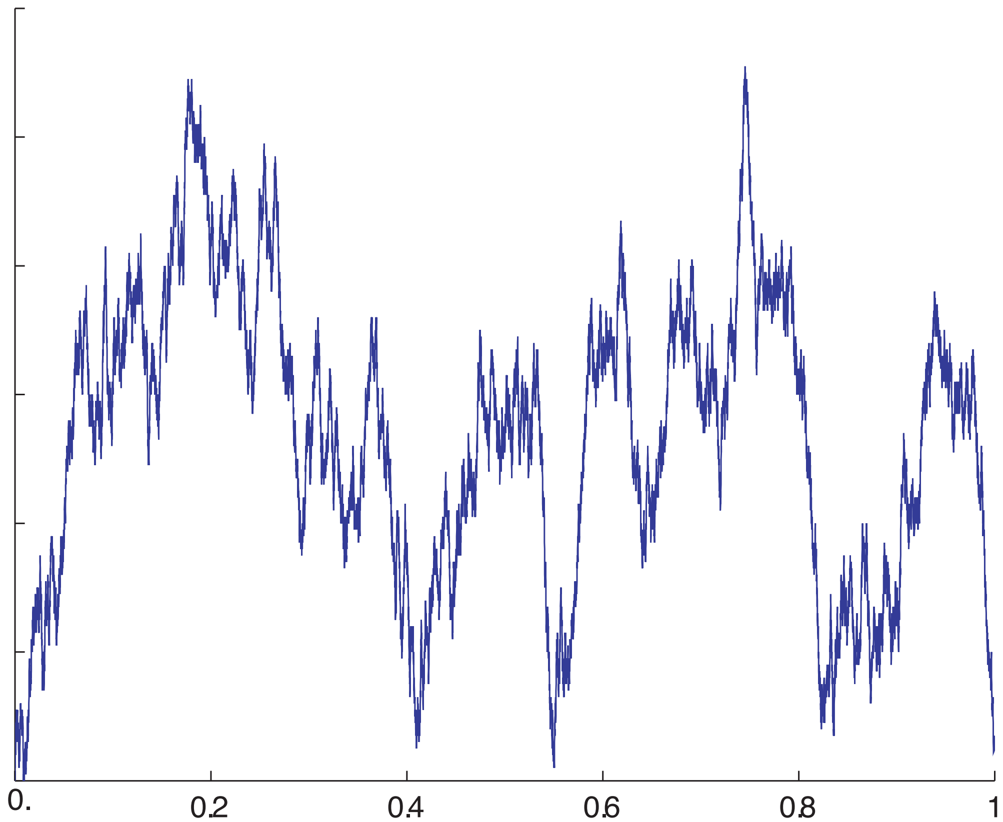

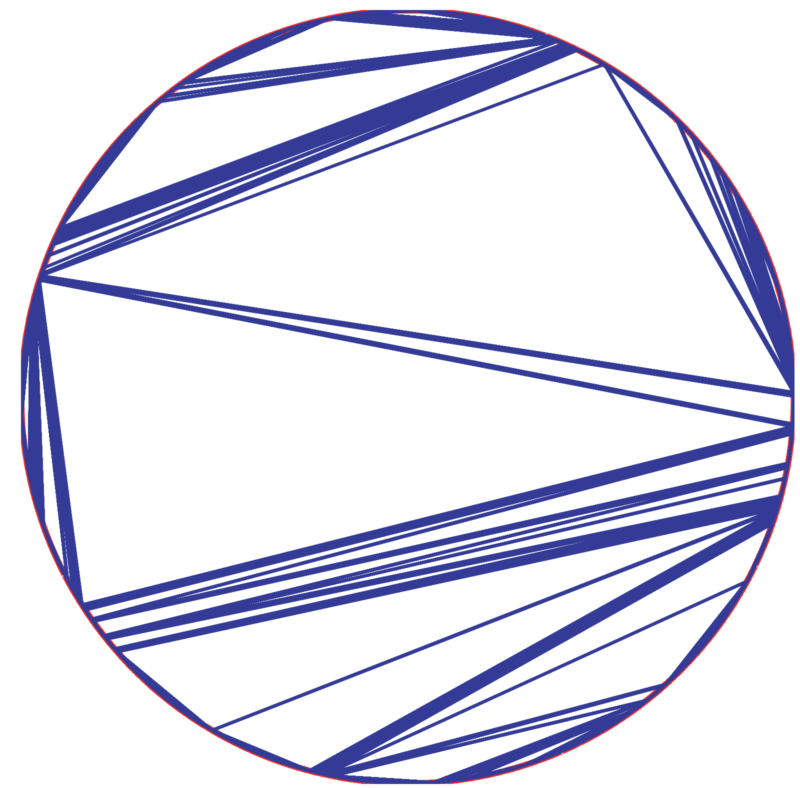

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The closure of this union, $L(\mathbb{e})$, is called the Brownian triangulation.

III. Minimal factorizations

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$\leadsto$ Question: for n large, what does a typical minimal factorization look like?

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$\rightsquigarrow$ Idea: compact subsets of the unit disk.

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
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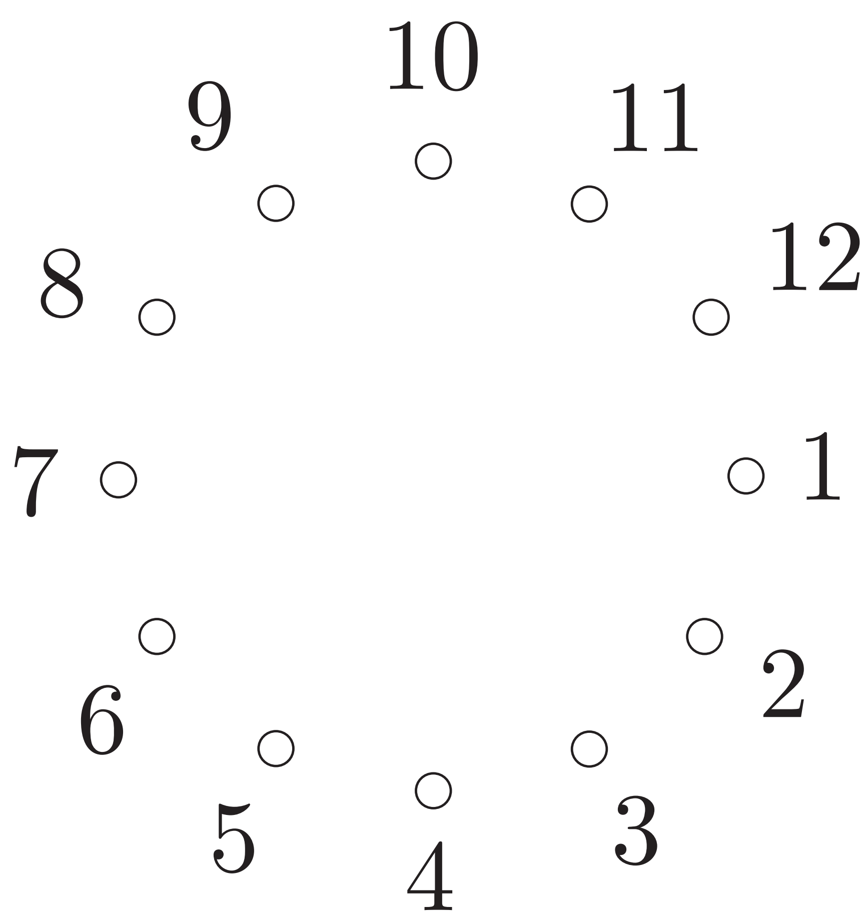

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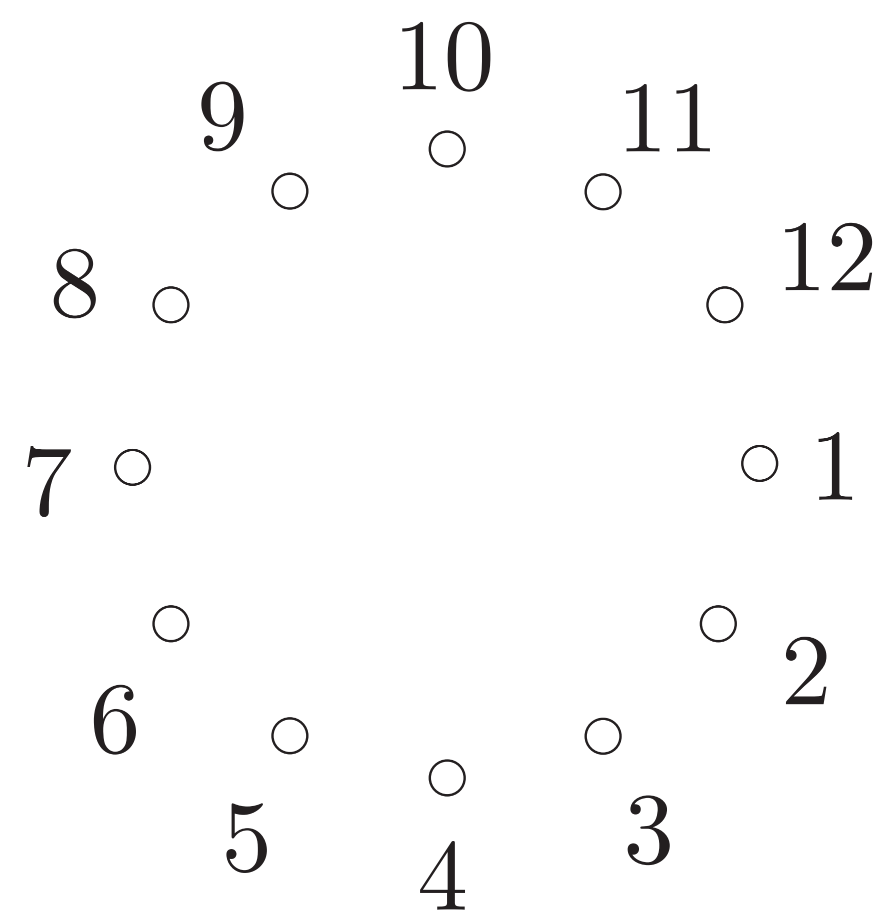

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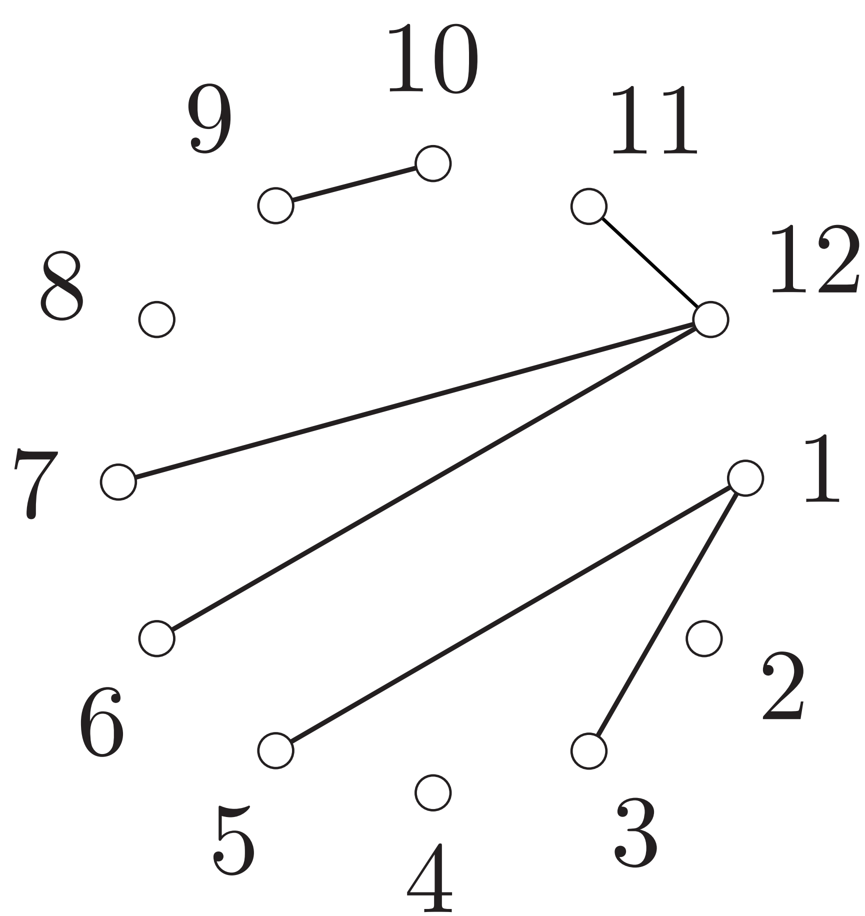

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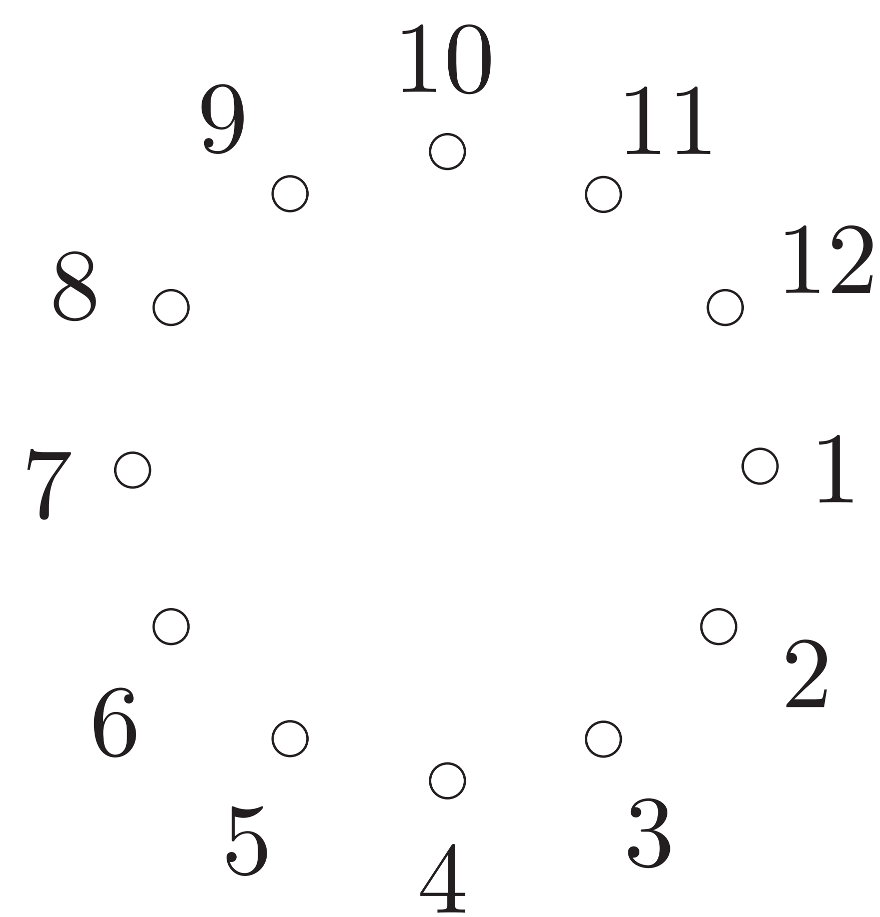

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If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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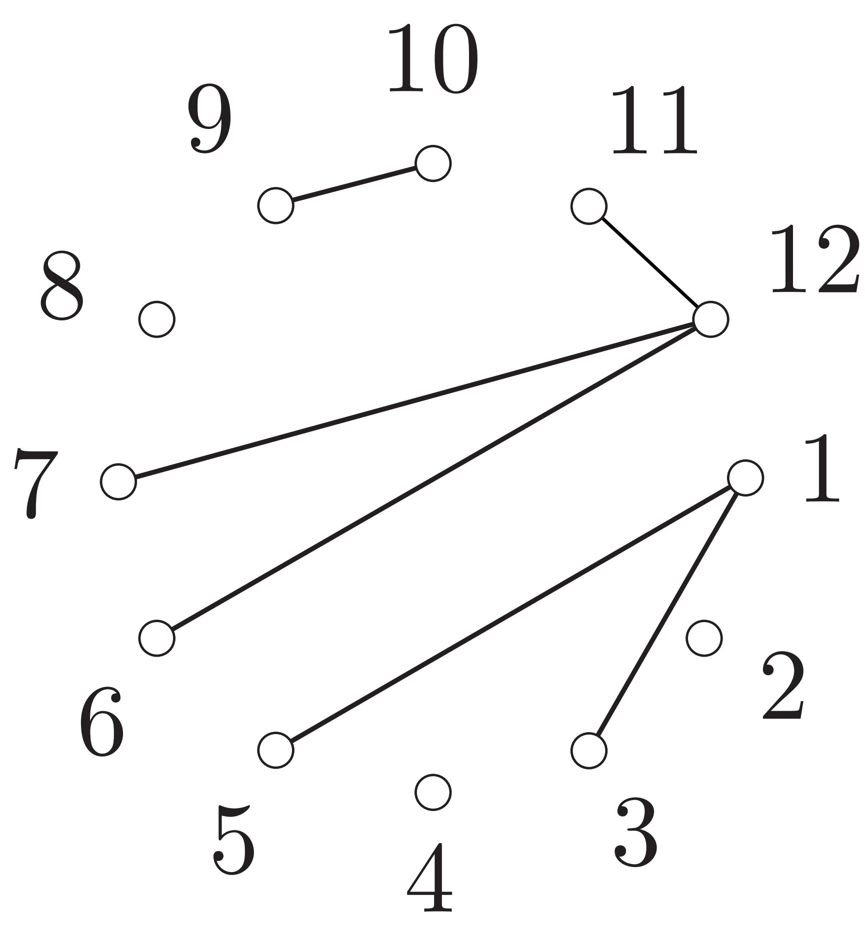
⇝ Example for $n = 12$ and $k = 6$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12)}_{\text{product}=(1,3,5)(6,7,11,12),(9,10)}, (2,3),(4,5),(1,6),(8,11),(9,11) \Big)$$



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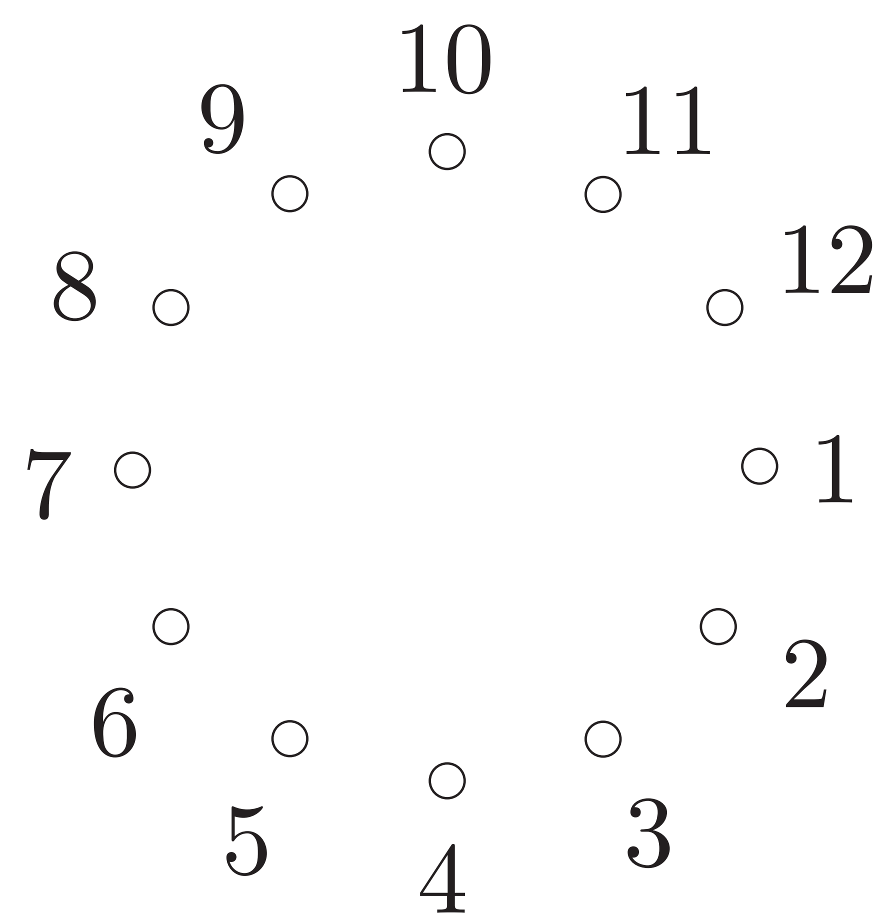

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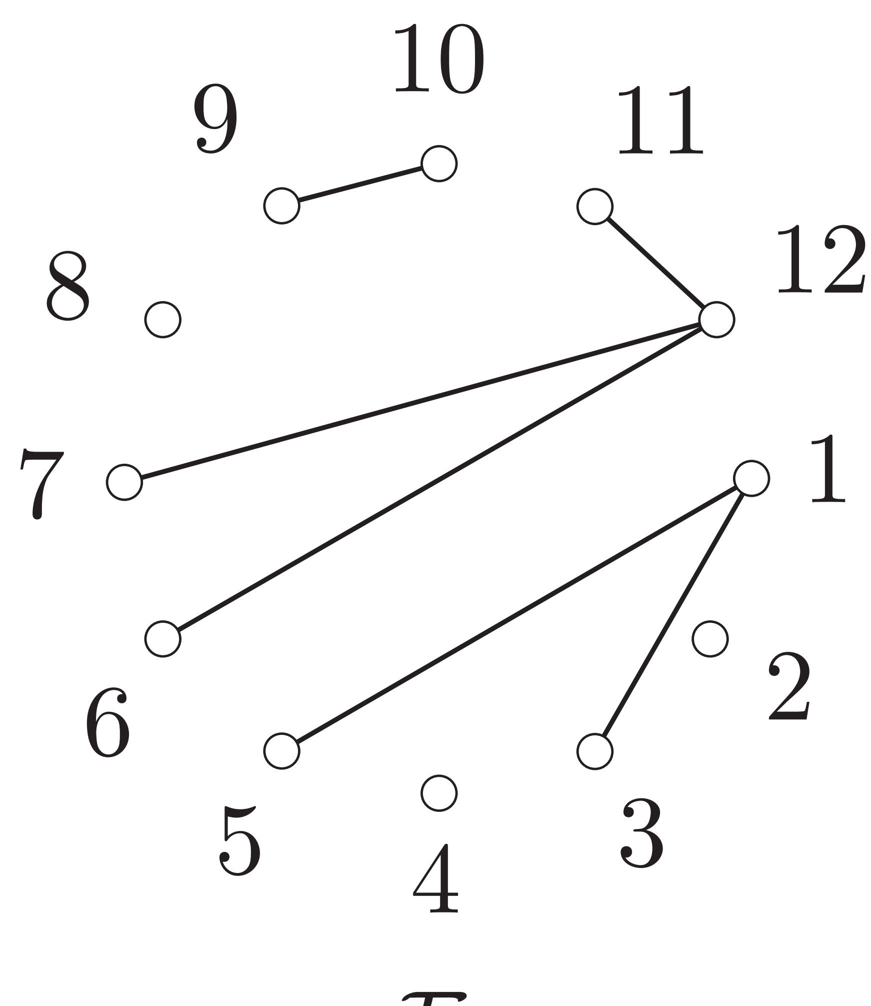

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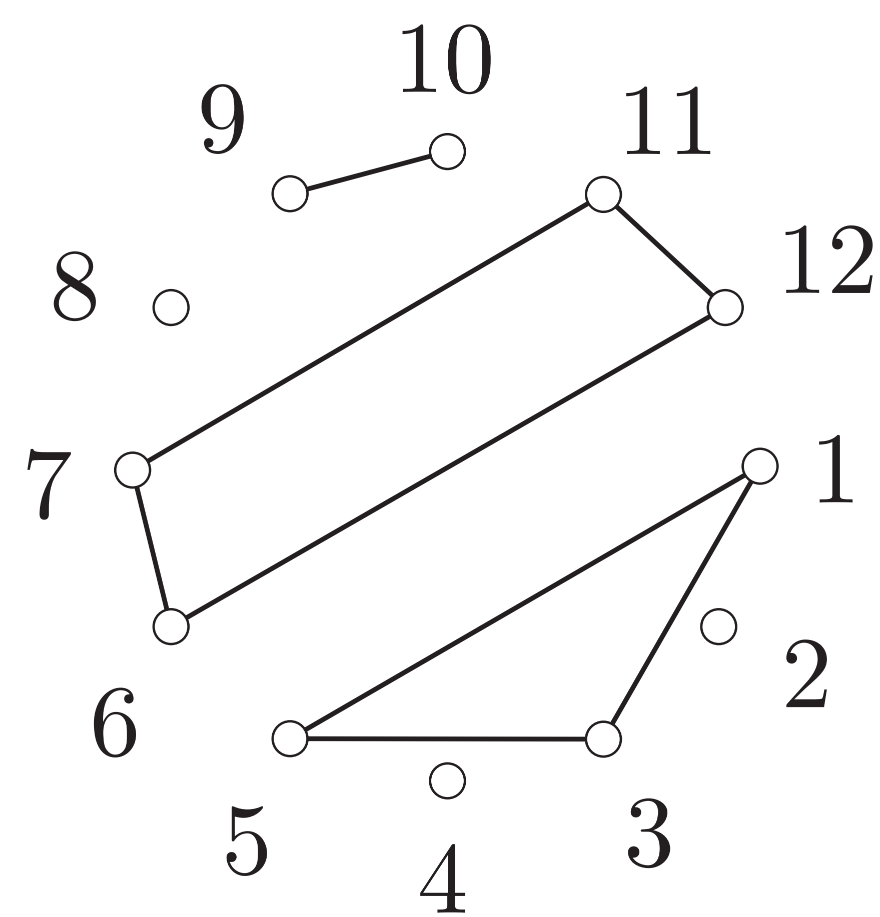

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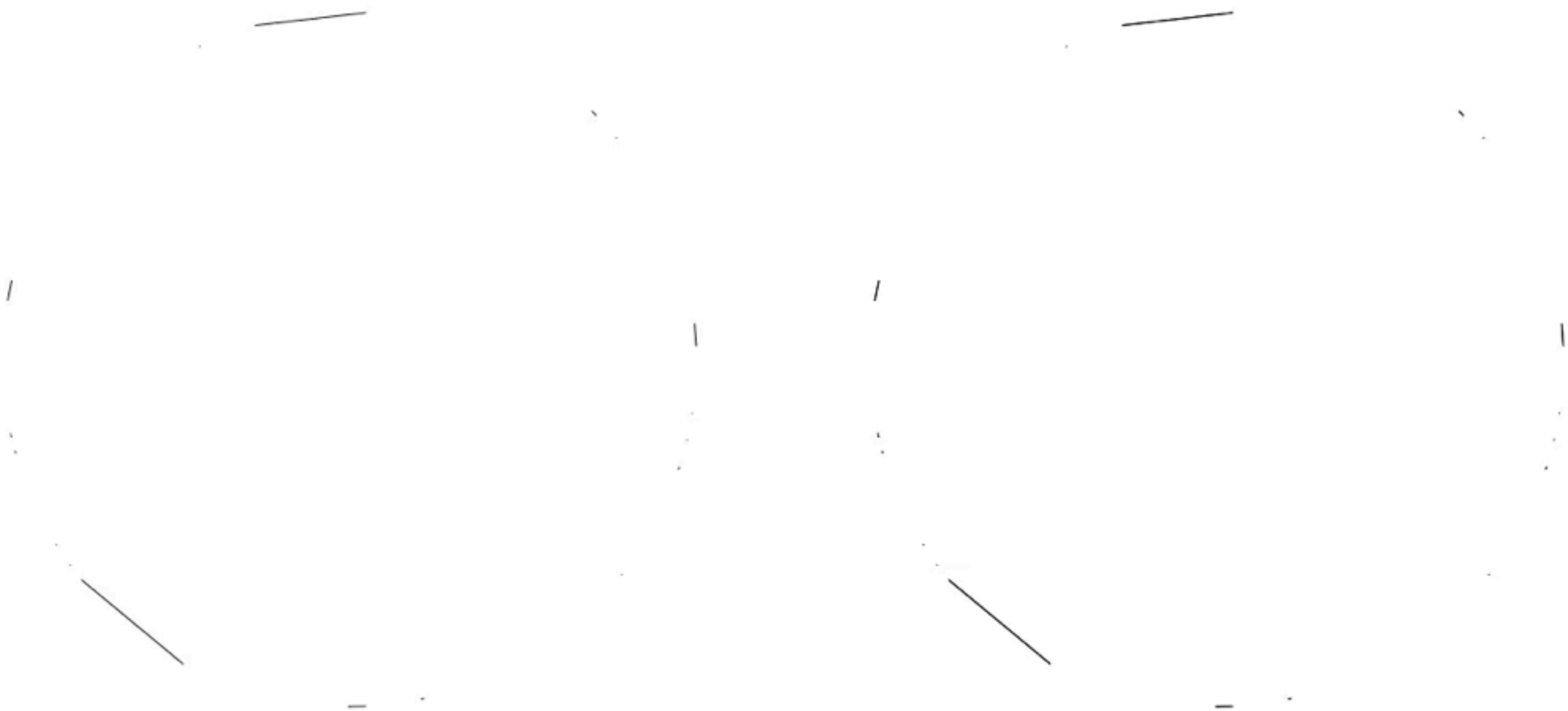
A simulation for $n = 2000$



Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

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(ii)

(iii)

(iv)

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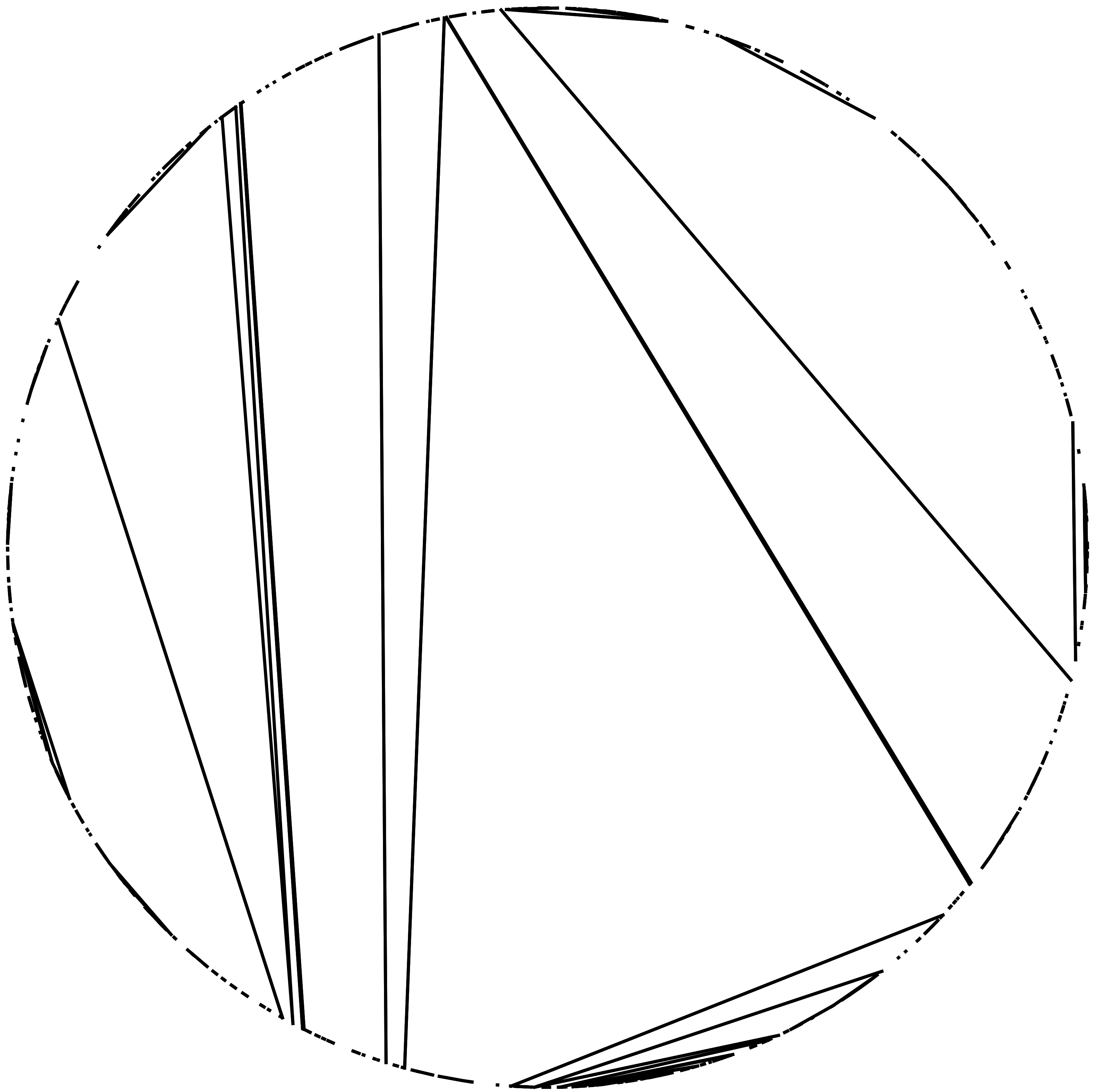

Figure: A simulation of $\mathbf{L}_5$.

Key fact

Proposition.

Fix $1 \leqslant k \leqslant n-1$ and let P be a non-crossing partition with n vertices and $n-k$ blocks. Then

$$\mathbb{P}\left(\mathcal{P}(t_1^{(n)} t_2^{(n)} \cdots t_k^{(n)}) = P\right)$$

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$$\mathbb{P}\left(t_1^{(n)} = (a, a+i)\right) = \frac{(n-2)!}{n^{n-2}} \cdot \frac{i^{i-2}}{(i-1)!} \cdot \frac{(n-i)^{(n-i-2)}}{(n-i-1)!}$$

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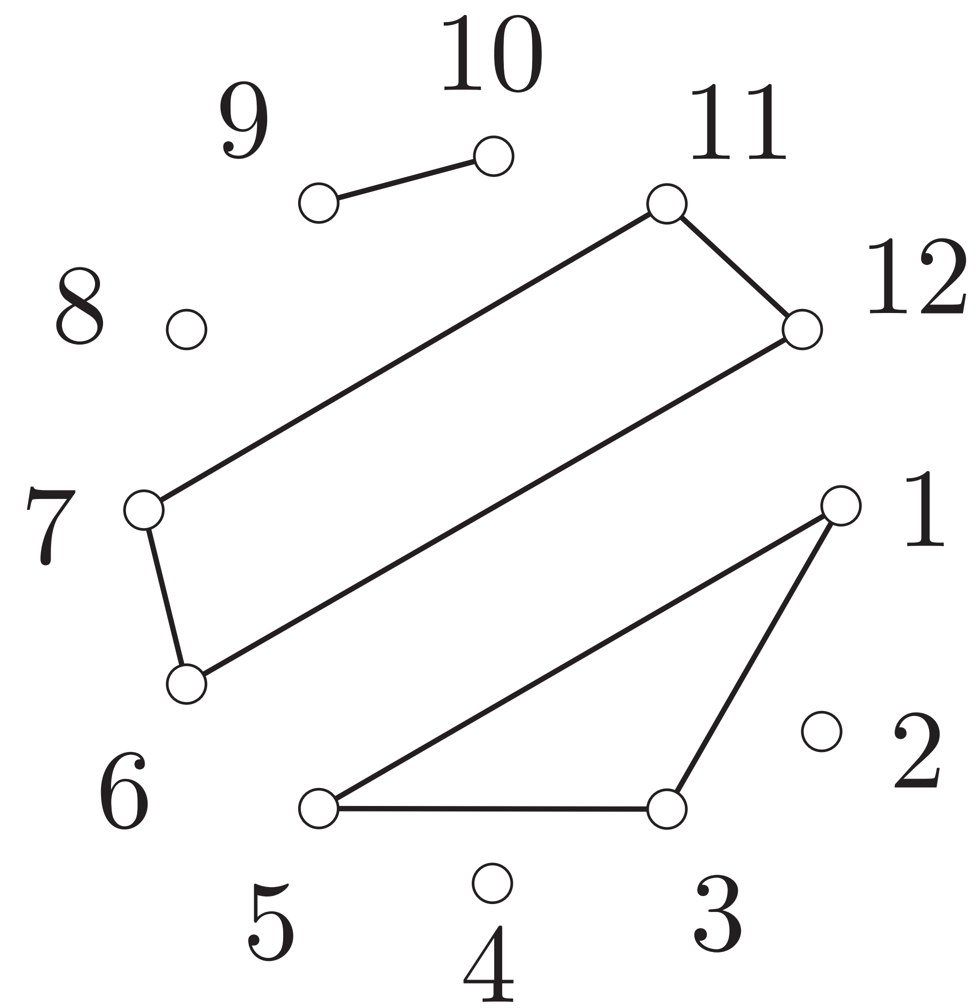
for n and i large, which explains the $\sqrt{n}$ transition.

⇝ Consequence 2:



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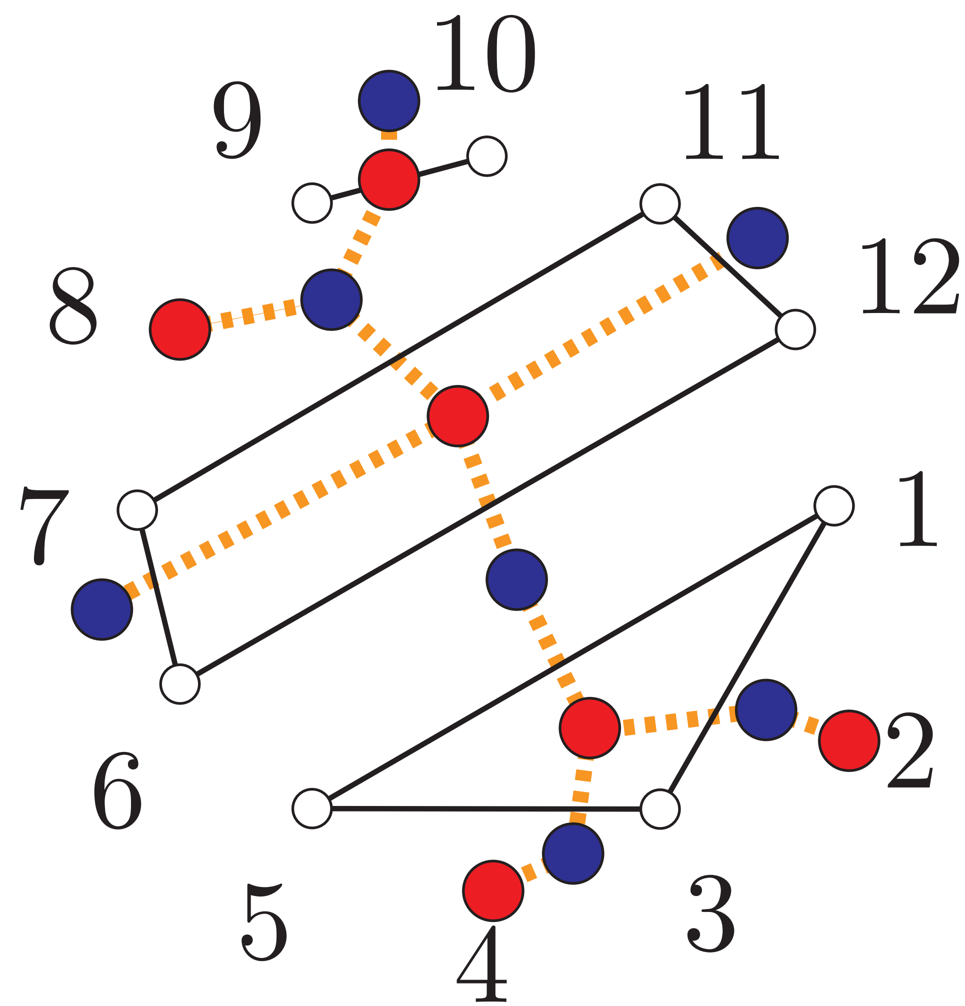

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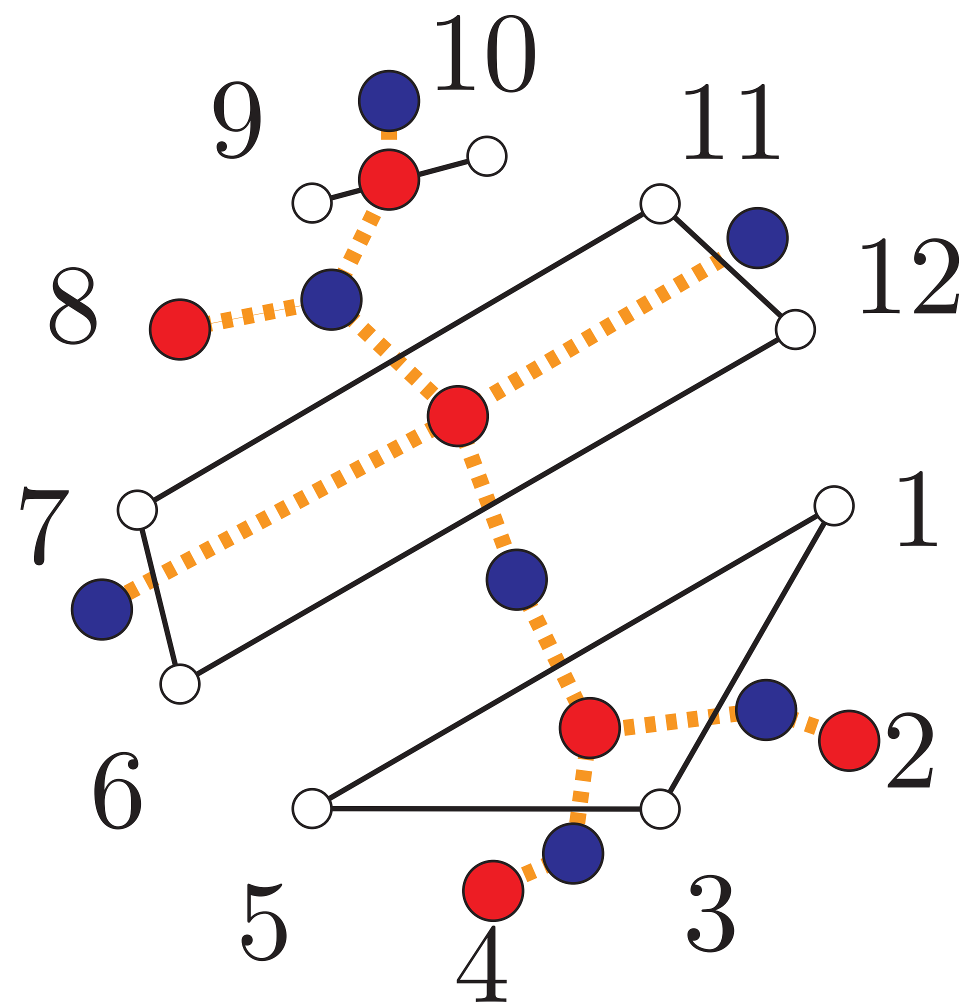

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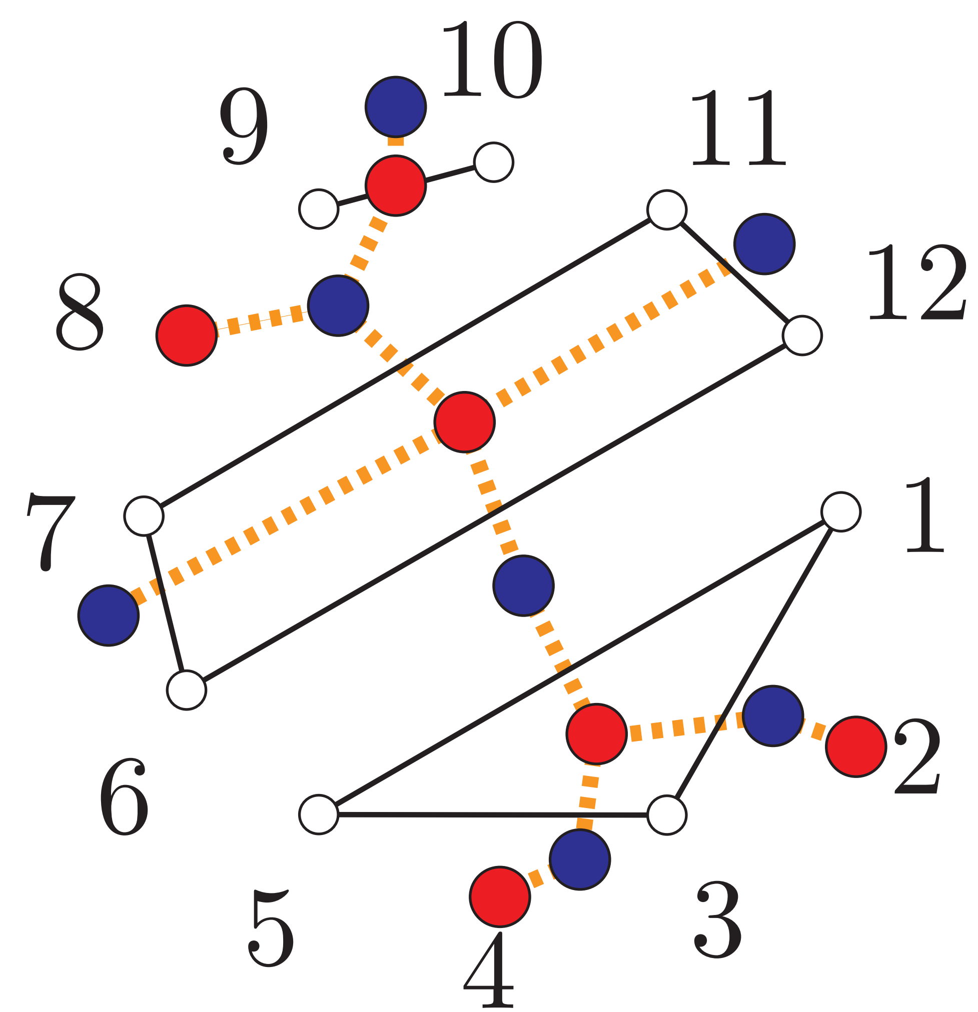

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(different conditioning than those considered for multitype BGW trees by Marckert, Miermont, Berzunza)

A proof of the half of (i), regime $K_n = o(\sqrt{n})$.

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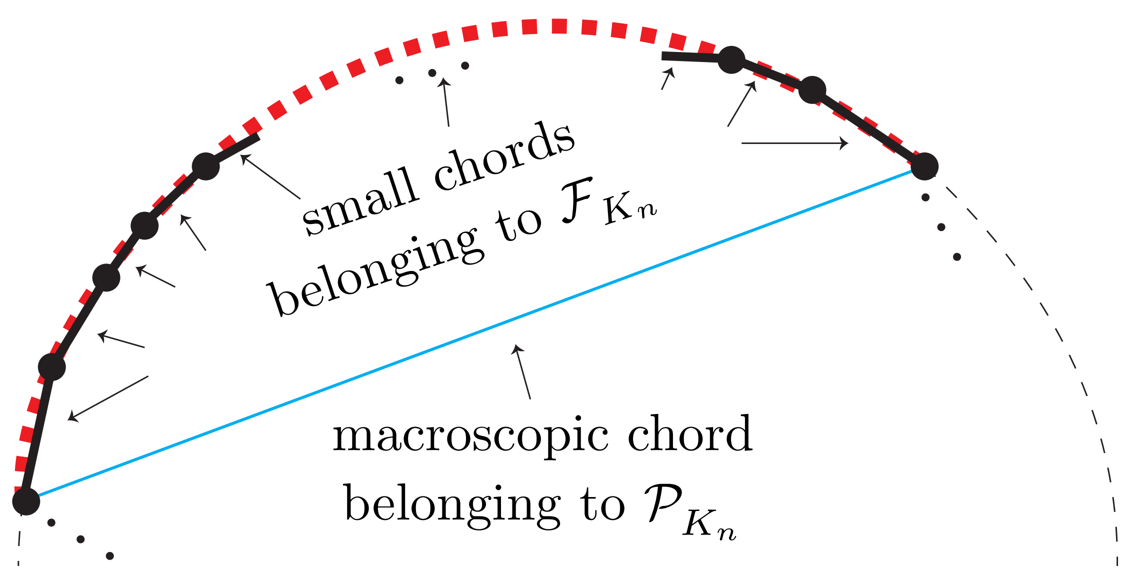

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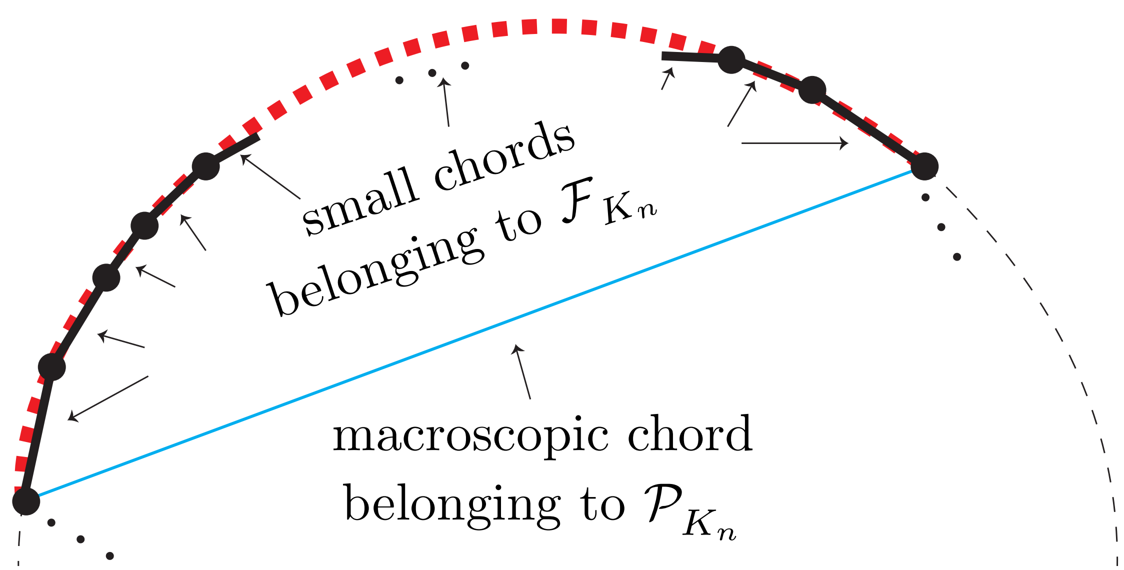

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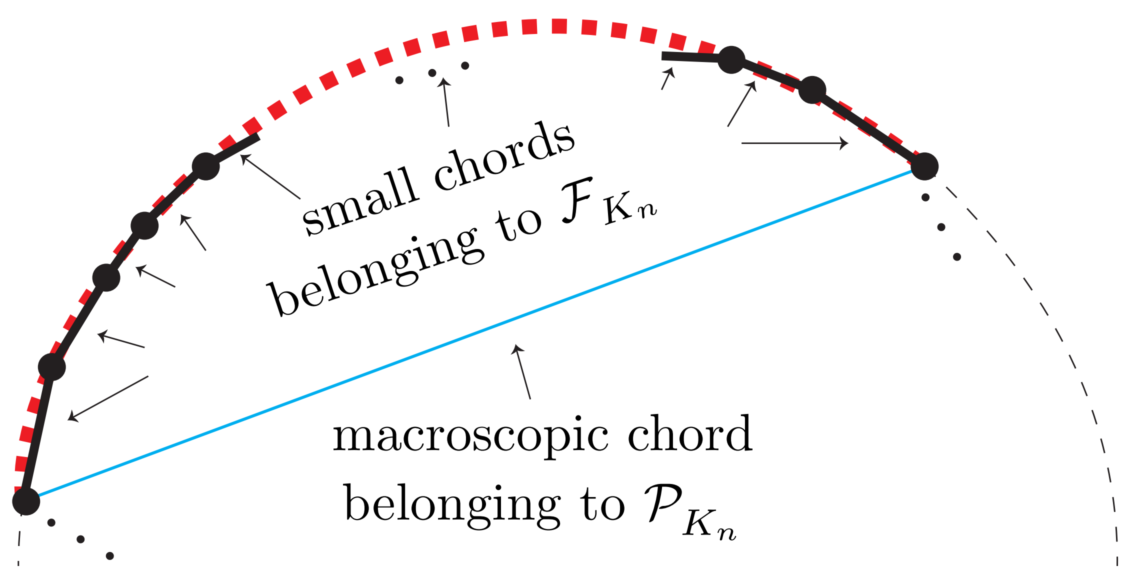

If this happens, there will be no macroscopic chord with endpoints in the red region in $\mathcal{F}_{n-1}$.

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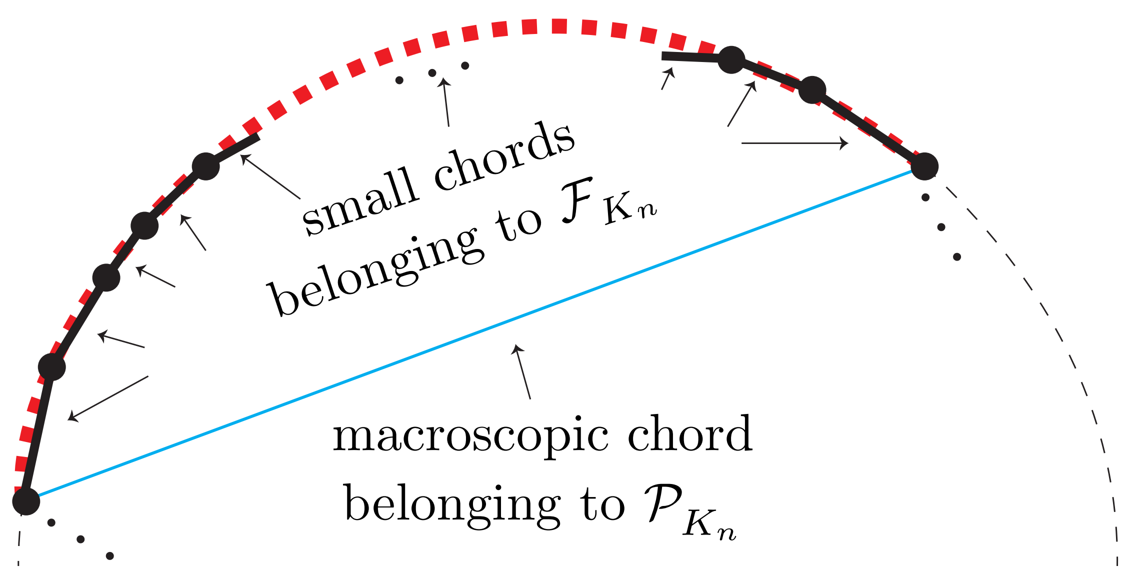

If this happens, there will be no macroscopic chord with endpoints in the red region in $\mathcal{F}_{n-1}$. This cannot happen in the Brownian triangulation.