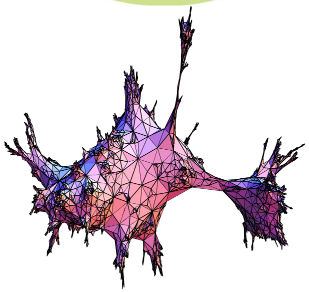
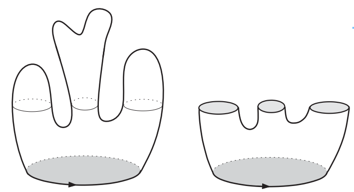
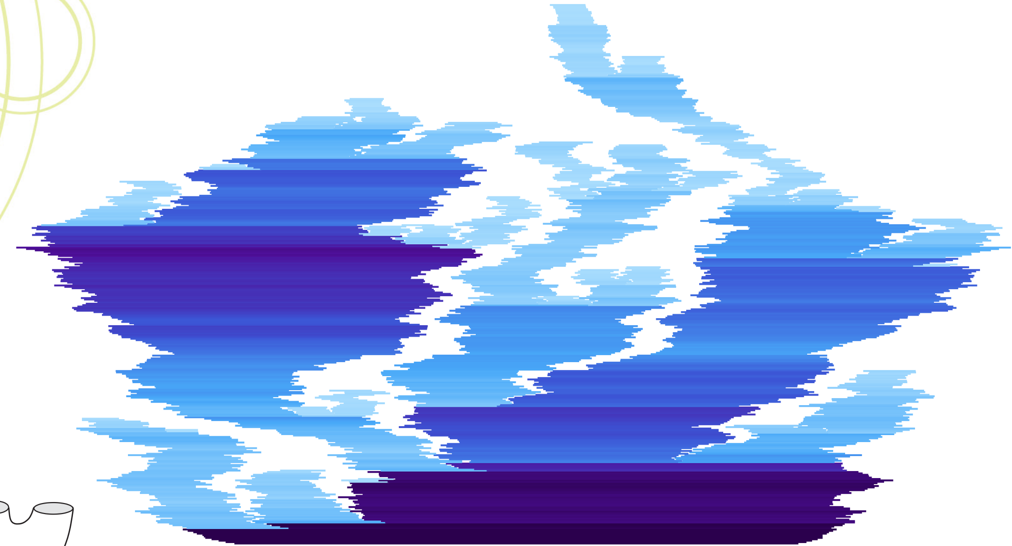






Random planar maps & growth-fragmentations

Igor Kortchemski
(joint work with J. Bertoin and N. Curien)

CNRS & École polytechnique

38th Conference on Stochastic Processes and their Applications
Spa2015@oxford-man.ox.ac.uk

Motivation

What does a “typical” random surface look like?

⇝ Idea: construct a (two-dimensional) random surface as a limit of random discrete surfaces.

$\rightsquigarrow$ Idea: construct a (two-dimensional) random surface as a limit of random discrete surfaces.

Consider n triangles, and glue them uniformly at random in such a way to get a surface homeomorphic to a sphere.

⤳ Idea: construct a (two-dimensional) random surface as a limit of random discrete surfaces.

Consider n triangles, and glue them uniformly at random in such a way to get a surface homeomorphic to a sphere.

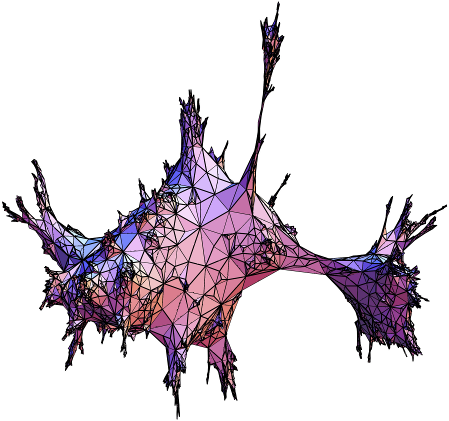


Figure: A large random triangulation (simulation by Nicolas Curien)

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles.

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance.

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map)

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map), in distribution for the Gromov–Hausdorff topology.

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map), in distribution for the Gromov–Hausdorff topology.

Solved by Le Gall in 2011.

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map), in distribution for the Gromov–Hausdorff topology.

Solved by Le Gall in 2011.

Since, many different models of discrete surfaces have been shown to converge to the Brownian map (Miermont, Beltran & Le Gall, Addario-Berry & Albenque, Bettinelli & Jacob & Miermont, Abraham)

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map), in distribution for the Gromov–Hausdorff topology.

Solved by Le Gall in 2011.

Since, many different models of discrete surfaces have been shown to converge to the Brownian map (Miermont, Beltran & Le Gall, Addario-Berry & Albenque, Bettinelli & Jacob & Miermont, Abraham), using various techniques (in particular bijective codings by labelled trees).

The Brownian map

Problem (Schramm at ICM '06): Let T_n be a random uniform triangulation of the sphere with n triangles. View T_n as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map), in distribution for the Gromov–Hausdorff topology.

Solved by Le Gall in 2011.

Since, many different models of discrete surfaces have been shown to converge to the Brownian map (Miermont, Beltran & Le Gall, Addario-Berry & Albenque, Bettinelli & Jacob & Miermont, Abraham), using various techniques (in particular bijective codings by labelled trees).

(see Le Gall's proceeding at ICM '14 for more information and references)

⇝ Other motivations:

– links with two dimensional Liouville Quantum Gravity (David, Duplantier, Garban, Kupianen, Maillard, Miller, Rhodes, Sheffield, Vargas, Zeitouni) c.f. the talks of Jason Miller, Scott Sheffield and Vincent Vargas.

– study of random planar maps decorated with statistical physics models (Angel, Berestycki, Borot, Bouttier, Guitter, Chen, Curien, Gwynne, K., Laslier, Mao, Ray, Sheffield, Sun, Wilson), c.f. the talk by Gourab Ray.

Outline

I. Boltzmann triangulations with a boundary

II. Peeling explorations

III. Cycles & growth-fragmentations

I. Boltzmann triangulations with a boundary

II. Peeling explorations

III. Cycles & growth-fragmentations

Triangulations

Definitions

A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations).

Definitions

A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations).

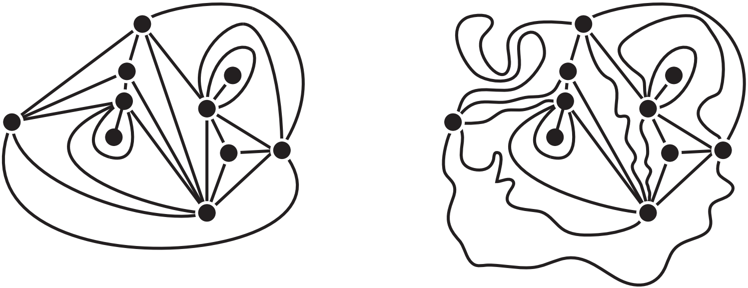


Figure: Two identical triangulations.

Definitions

A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations). A map is a **triangulation** when all the faces are triangles.

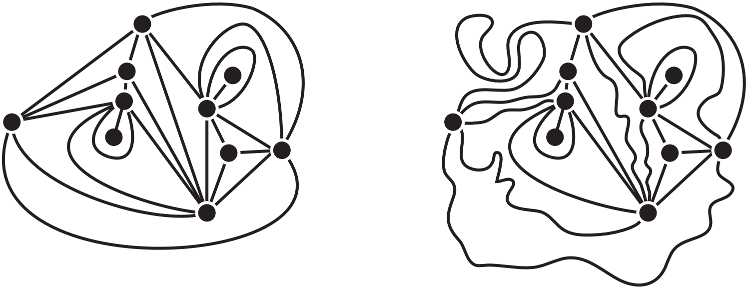


Figure: Two identical triangulations.

Definitions

A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations). A map is a **triangulation** when all the faces are triangles. A map is **rooted** when an oriented edge is distinguished.

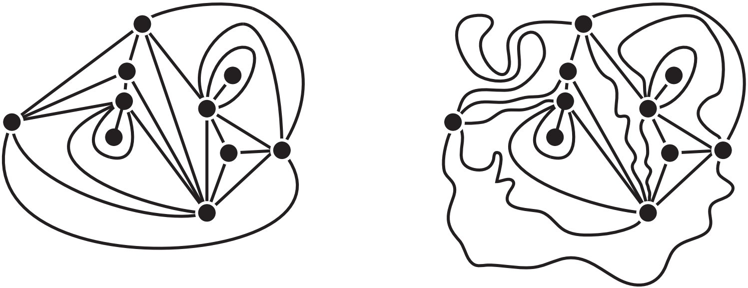


Figure: Two identical triangulations.

Definitions

A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations). A map is a **triangulation** when all the faces are triangles. A map is **rooted** when an oriented edge is distinguished.

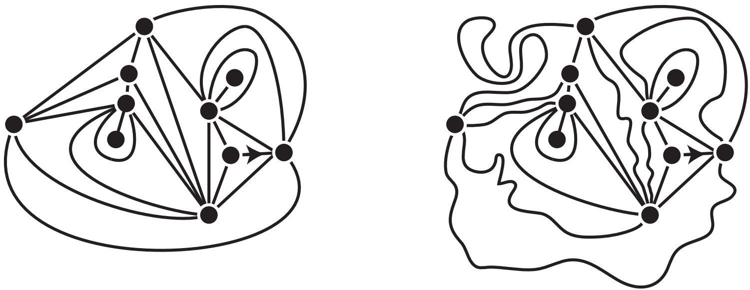
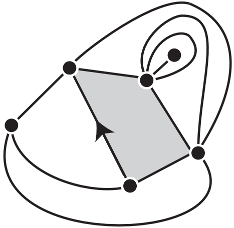


Figure: Two identical rooted triangulations.

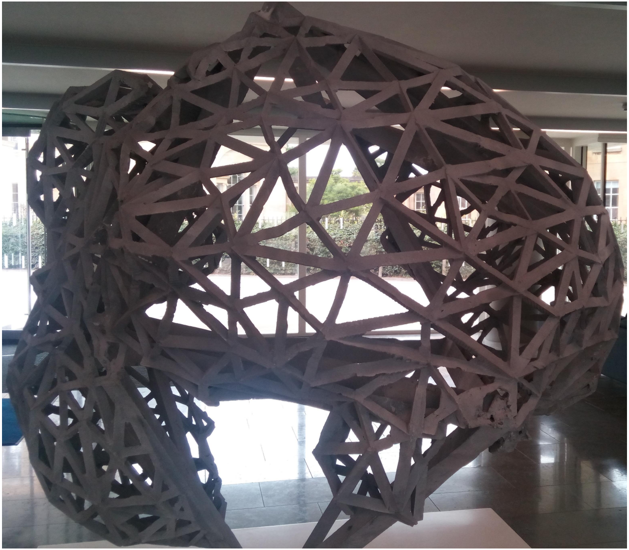
Triangulations with a boundary

Definitions

A **triangulation with a boundary** is a map where all faces are triangles, except possibly the one to the right of the root edge, called the **external face**.

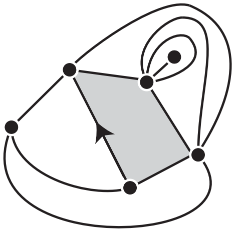


Another example of a triangulation with a boundary



Definitions

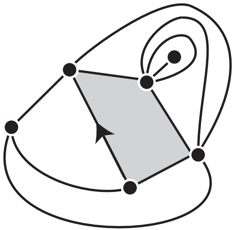
A **triangulation with a boundary** is a map where all faces are triangles, except possibly the one to the right of the root edge, called the **external face**.



A **triangulation of the p-gon** is a triangulation with a simple boundary of length p.

Definitions

A **triangulation with a boundary** is a map where all faces are triangles, except possibly the one to the right of the root edge, called the **external face**.



A **triangulation of the p-gon** is a triangulation with a simple boundary of length p.

A triangulation of the p-gon chosen at random proportionally to

$$(12\sqrt{3})^{-\#(\text{internal vertices})}$$

is called a (critical) **Boltzmann triangulation of the p-gon**.

Cycles at heights

The goal

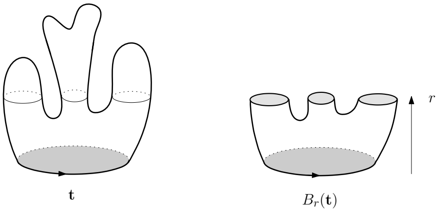
Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon

The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(T^{(p)})$ its ball of radius r

The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(T^{(p)})$ its ball of radius r

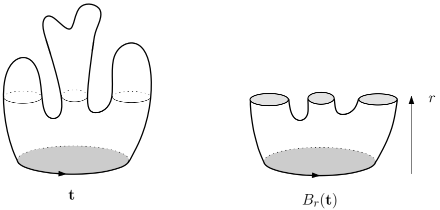


The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(T^{(p)})$ its ball of radius r, and

$$\mathbf{L}^{(p)}(r) := \left(L_1^{(p)}(r), L_2^{(p)}(r), \dots \right).$$

be the lengths (or perimeters) of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

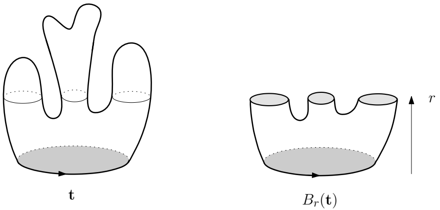


The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(T^{(p)})$ its ball of radius r, and

$$\mathbf{L}^{(p)}(r) := \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right).$$

be the lengths (or perimeters) of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.



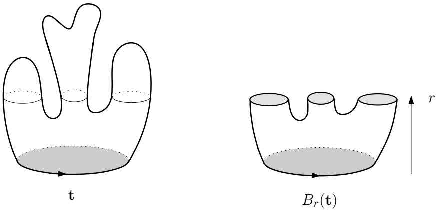
⇝ Goal: obtain a functional invariance principle for $(\mathbf{L}^{(p)}(r); r \geqslant 0)$.

The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(T^{(p)})$ its ball of radius r, and

$$\mathbf{L}^{(p)}(r) := \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots \right).$$

be the lengths (or perimeters) of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.



⇝ Goal: obtain a functional invariance principle for $(\mathbf{L}^{(p)}(r); r \geqslant 0)$. In this spirit, a "breadth-first search" description of the Brownian map is given by Miller & Sheffield '15.

I. Boltzmann triangulations with a boundary

II. Peeling explorations

III. Cycles & growth-fragmentations

Geometry of random maps

Several techniques to study random maps:

Geometry of random maps

Several techniques to study random maps:

– **bijective techniques**, following the work of Schaeffer '98.

Geometry of random maps

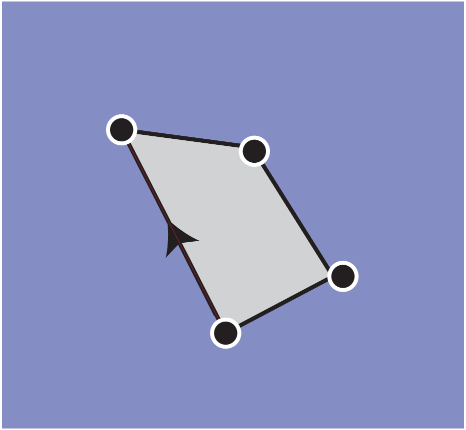
Several techniques to study random maps:

– **bijective techniques**, following the work of Schaeffer '98.

– **peeling process**, which is an algorithmic procedure that explores a map step-by-step in a Markovian way (Watabiki '95, Angel '03).

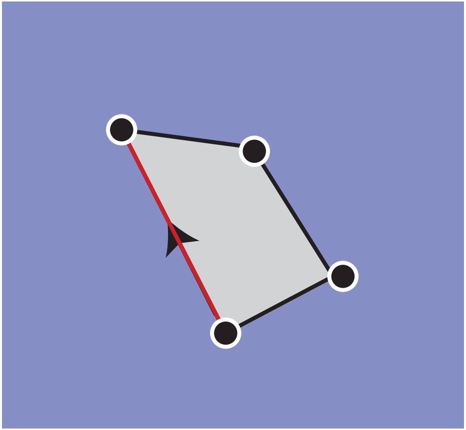
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



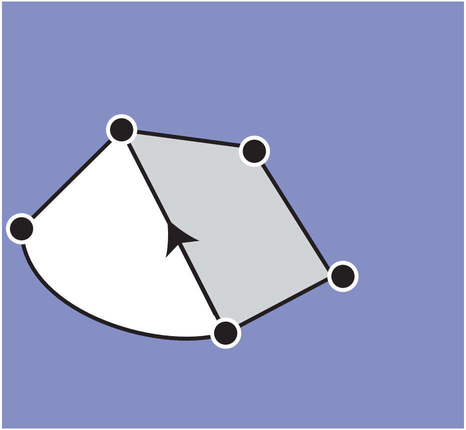
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



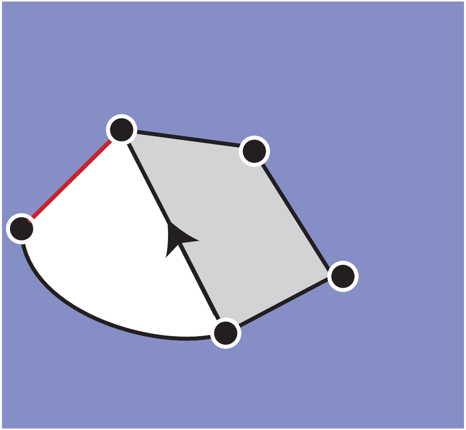
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



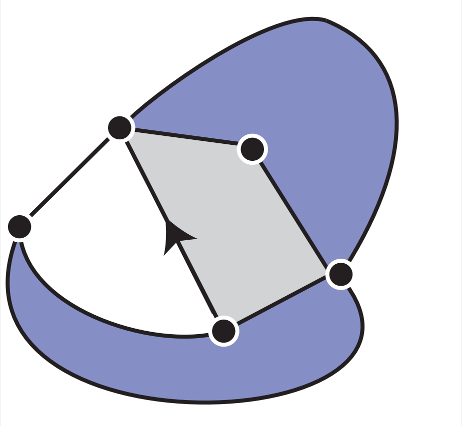
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



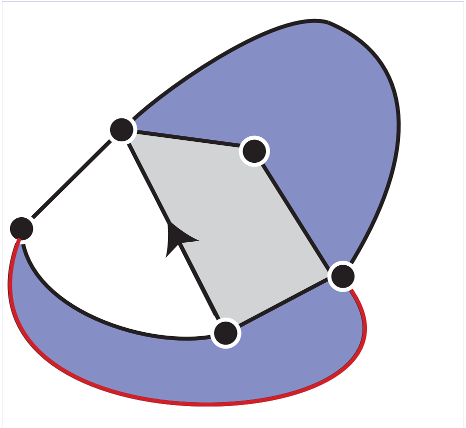
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



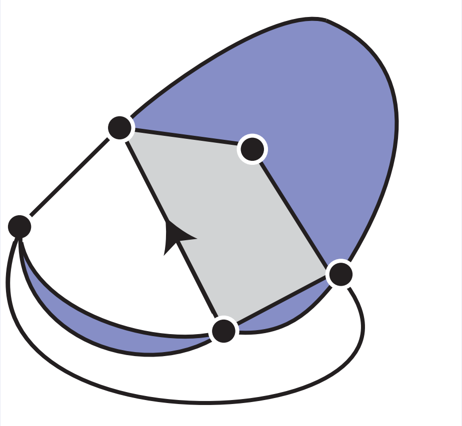
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



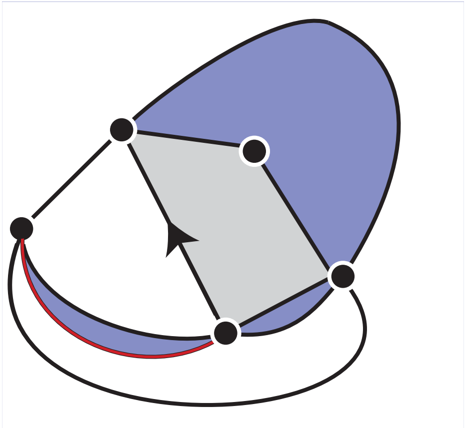
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



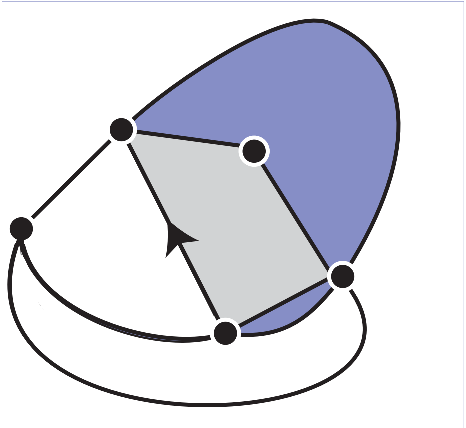
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



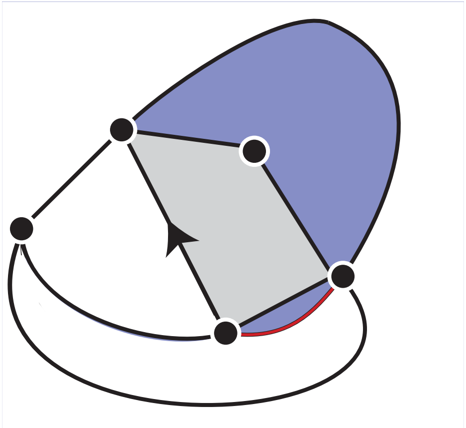
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



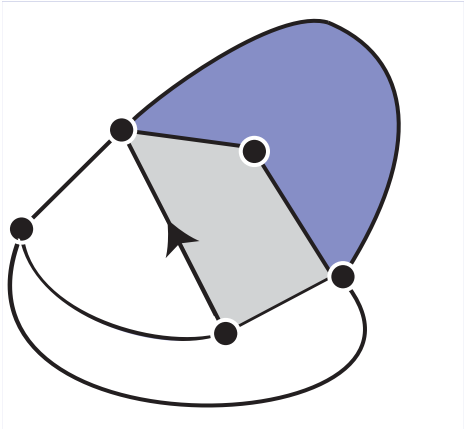
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



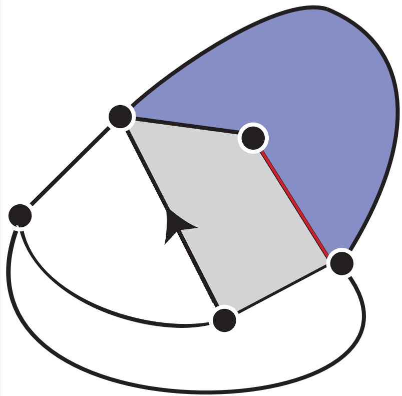
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



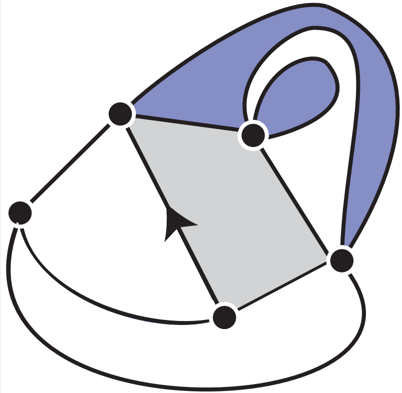
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



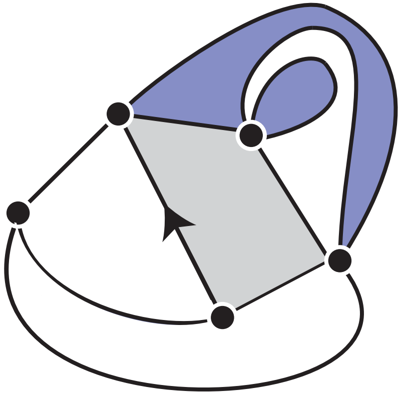
Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



Branching peeling explorations

Intuitively speaking, the **branching peeling process** of a triangulation $\mathfrak{t}$ is a way to iteratively explore $\mathfrak{t}$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$.



And so on...

I. Boltzmann triangulations with a boundary

II. Peeling explorations

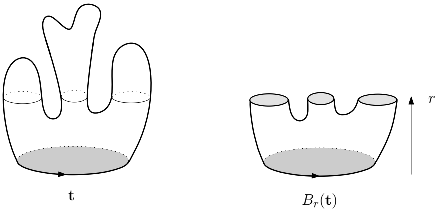
III. Cycles & growth-fragmentations

The goal

Let $\mathsf{T}^{(p)}$ be a random Boltzmann triangulation of the p-gon, $B_r(\mathsf{T}^{(p)})$ its ball of radius r, and

$$\mathbf{L}^{(p)}(r) := \left(L_1^{(p)}(r), L_2^{(p)}(r), \dots\right).$$

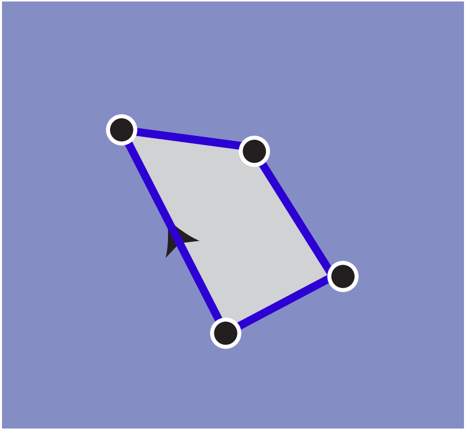
be the lengths (or perimeters) of the cycles of $B_r(\mathsf{T}^{(p)})$ ranked in decreasing order.



$\leadsto$ Goal: obtain a functional invariance principle for $(\mathbf{L}^{(p)}(r); r \geqslant 0)$.

Following the locally largest cycle

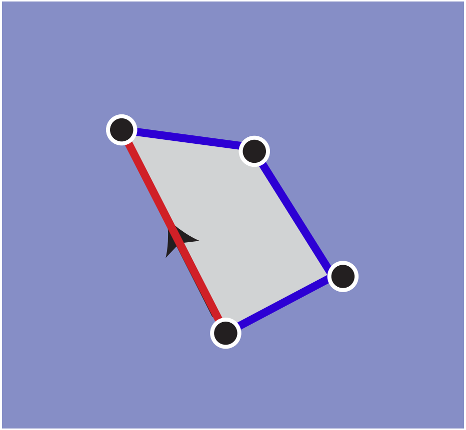
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$

Following the locally largest cycle

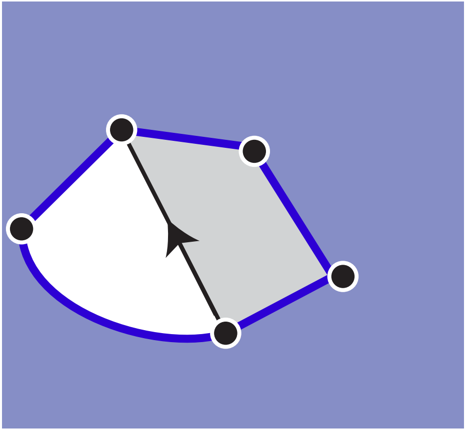
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$

Following the locally largest cycle

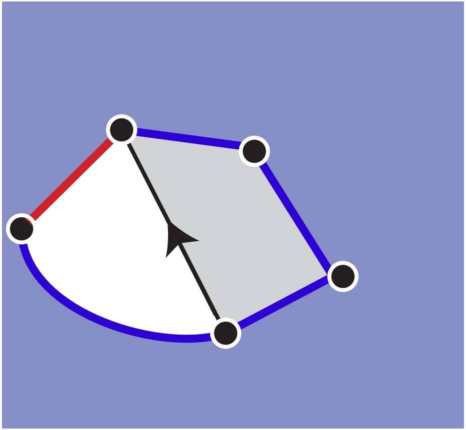
$\rightsquigarrow$ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$

Following the locally largest cycle

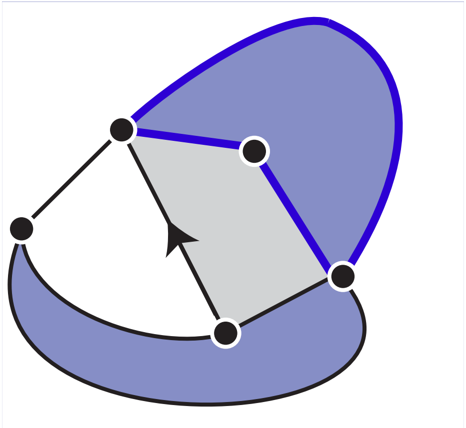
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$

Following the locally largest cycle

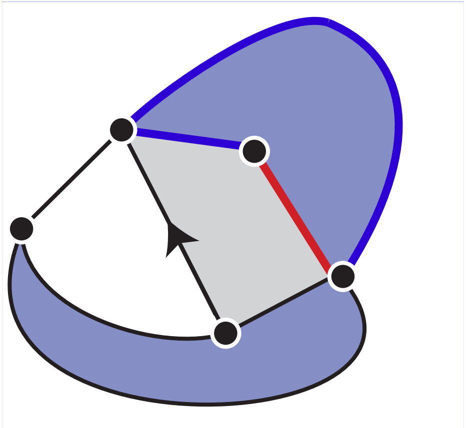
$\rightsquigarrow$ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$

Following the locally largest cycle

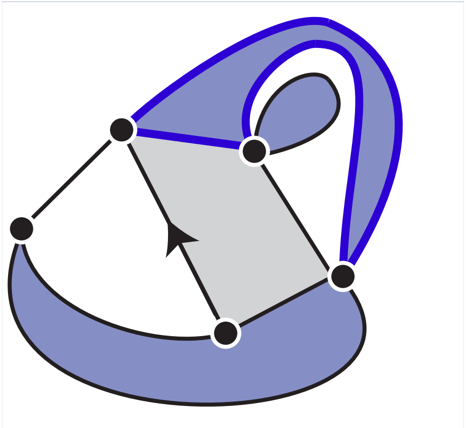
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$

Following the locally largest cycle

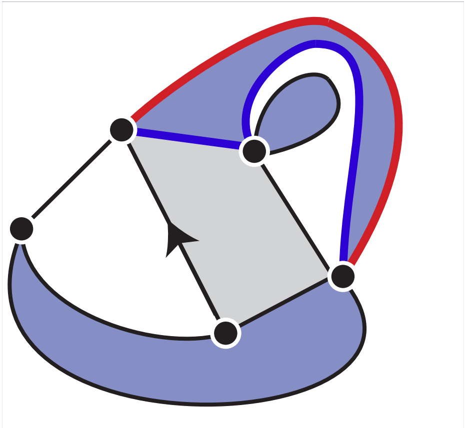
↝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$, $\widetilde{L}^{(4)}(3) = 3$

Following the locally largest cycle

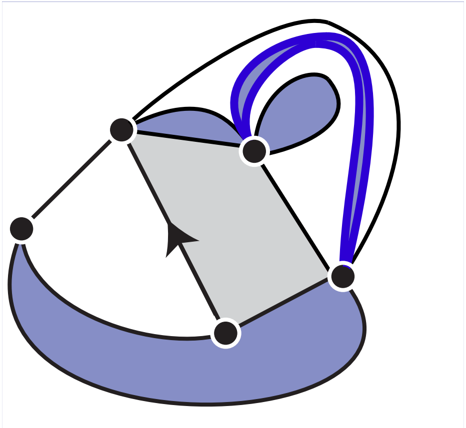
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$, $\widetilde{L}^{(4)}(3) = 3$

Following the locally largest cycle

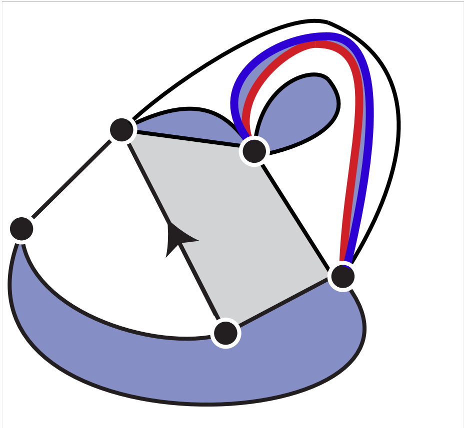
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$, $\widetilde{L}^{(4)}(3) = 3$, $\widetilde{L}^{(4)}(4) = 2$

Following the locally largest cycle

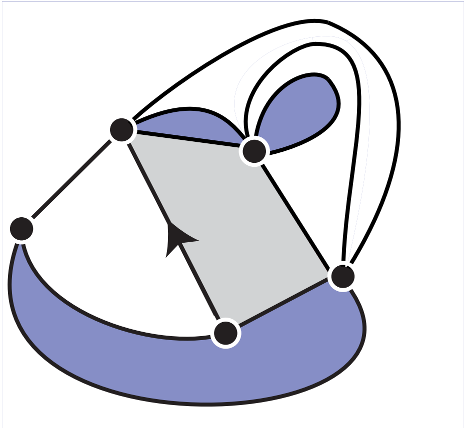
⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$, $\widetilde{L}^{(4)}(3) = 3$, $\widetilde{L}^{(4)}(4) = 2$

Following the locally largest cycle

⇝ Idea: follow the locally largest cycle at each peeling step and consider its length $\widetilde{L}^{(p)}(r)$ after r peeling steps.



$\widetilde{L}^{(4)}(0) = 4$, $\widetilde{L}^{(4)}(1) = 5$, $\widetilde{L}^{(4)}(2) = 3$, $\widetilde{L}^{(4)}(3) = 3$, $\widetilde{L}^{(4)}(4) = 2$, $\widetilde{L}^{(4)}(5) = 0$.

Scaling limit for the locally largest cycle

Recall that $\widetilde{L}^{(p)}(r)$ the length of the locally largest cycle after r peeling steps.

Scaling limit for the locally largest cycle

Recall that $\widetilde{L}^{(p)}(r)$ the length of the locally largest cycle after r peeling steps.

$\rightsquigarrow$ Key fact : $(\widetilde{L}^{(p)}(r); r \geqslant 0)$ is a Markov chain on the nonnegative integers, started at p, absorbed at 0 and with explicit transitions.

Scaling limit for the locally largest cycle

Recall that $\widetilde{L}^{(p)}(r)$ the length of the locally largest cycle after r peeling steps.

$\rightsquigarrow$ Key fact : $(\widetilde{L}^{(p)}(r); r \geqslant 0)$ is a Markov chain on the nonnegative integers, started at p, absorbed at 0 and with explicit transitions. In addition, the triangulations filling-in the unexplored holes are Boltzmann triangulations.

Scaling limit for the locally largest cycle

Recall that $\widetilde{L}^{(p)}(r)$ the length of the locally largest cycle after r peeling steps.

⇝ Key fact : $(\widetilde{L}^{(p)}(r); r \geqslant 0)$ is a Markov chain on the nonnegative integers, started at p, absorbed at 0 and with explicit transitions. In addition, the triangulations filling-in the unexplored holes are Boltzmann triangulations.

If $L^{(p)}(r)$ the length of the locally largest cycle at height r

Scaling limit for the locally largest cycle

Recall that $\widetilde{L}^{(p)}(r)$ the length of the locally largest cycle after r peeling steps.

$\rightsquigarrow$ Key fact : $(\widetilde{L}^{(p)}(r); r \geqslant 0)$ is a Markov chain on the nonnegative integers, started at p, absorbed at 0 and with explicit transitions. In addition, the triangulations filling-in the unexplored holes are Boltzmann triangulations.

If $L^{(p)}(r)$ the length of the locally largest cycle at height r, with the help of Bertoin & K. '14 and Curien & Le Gall '14, we get that:

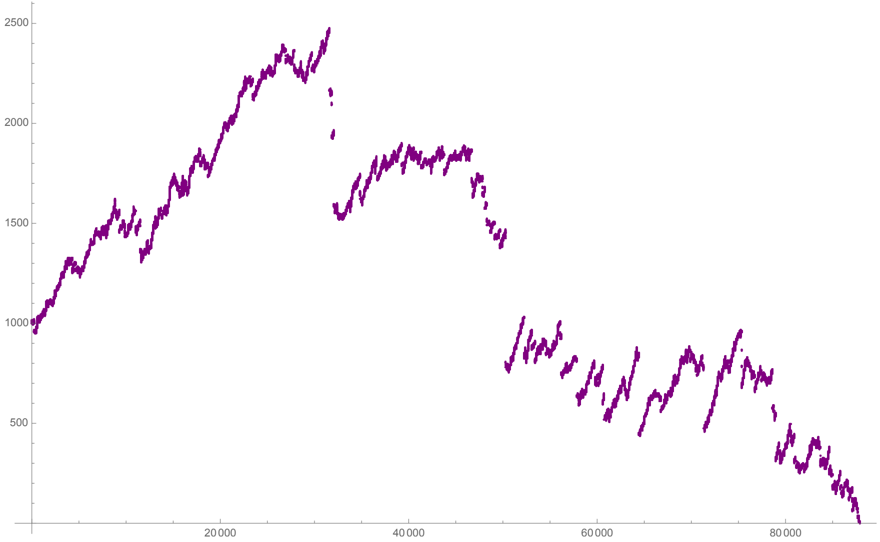
Proposition (Bertoin, Curien & K. '15).

We have

$$\left(\frac{1}{p} L^{(p)}\left(\lfloor \sqrt{p} \cdot t \rfloor\right); t \geqslant 0\right) \quad \xrightarrow[p \to \infty]{(d)} \quad \left(X\left(\frac{3}{2\sqrt{\pi}} \cdot t\right); t \geqslant 0\right),$$

where X is a càdlàg self-similar process with $X(0) = 1$ and absorbed at 0.

The self-similar process X



The self-similar process X

Let ξ be the spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left(x(1-x)\right)^{-5/2} dx,$$

so that $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geqslant 0$ and $q \geqslant 0$.

The self-similar process X

Let ξ be the spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left(x(1-x)\right)^{-5/2} dx,$$

so that $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geqslant 0$ and $q \geqslant 0$.

Finally, set

$$X(t) = \exp\left(\xi(\tau(t))\right), \qquad t \geqslant 0$$

The self-similar process X

Let ξ be the spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left(x(1-x)\right)^{-5/2} dx,$$

so that $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geqslant 0$ and $q \geqslant 0$.

Then set

$$\tau(t) = \inf\left\{u \geqslant 0; \int_0^u e^{\xi(s)/2} ds > t\right\}, \qquad t \geqslant 0$$

with the convention that $\inf \emptyset = \infty$, i.e. $\tau(t) = \infty$ whenever $t \geqslant \int_0^\infty e^{\xi(s)/2} ds$.

Finally, set

$$X(t) = \exp\left(\xi(\tau(t))\right), \qquad t \geqslant 0$$

The self-similar process X

Let ξ be the spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left(x(1-x)\right)^{-5/2} dx,$$

so that $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geqslant 0$ and $q \geqslant 0$.

Then set

$$\tau(t) = \inf \left\{ u \geqslant 0; \int_0^u e^{\xi(s)/2} ds > t \right\}, \qquad t \geqslant 0$$

with the convention that $\inf \emptyset = \infty$, i.e. $\tau(t) = \infty$ whenever $t \geqslant \int_0^\infty e^{\xi(s)/2} ds$.

Finally, set

$$X(t) = \exp\left(\xi(\tau(t))\right), \qquad t \geqslant 0$$

(with the convention $\exp(\xi(\infty)) = 0$), which is a self-similar Markov process (Lamperti transformation).

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations.

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X.

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

$\leadsto$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

⇝ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y)

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

$\leadsto$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y), and is independent of the processes of all the other daughter particles.

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

$\leadsto$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y), and is independent of the processes of all the other daughter particles.

And so on for the granddaughters, then great-granddaughters, ...

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

$\rightsquigarrow$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y), and is independent of the processes of all the other daughter particles.

And so on for the granddaughters, then great-granddaughters, ...

By Bertoin '15, for every $t \geqslant 0$, the family of the sizes of cells which are present in the system at time t is cube-summable

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

⇝ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y), and is independent of the processes of all the other daughter particles.

And so on for the granddaughters, then great-granddaughters, ...

By Bertoin '15, for every $t \geqslant 0$, the family of the sizes of cells which are present in the system at time t is cube-summable, and can therefore be ranked in non-increasing order.

Defining the growth-fragmentation

We use X to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age t, and:

– Start at time 0 from a single cell with size 1, and suppose that its size evolves according to X. We interpret each (negative) jump of X as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time t into a mother cell and a daughter.

$\leadsto$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size y and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process X (starting of course from y), and is independent of the processes of all the other daughter particles.

And so on for the granddaughters, then great-granddaughters, ...

By Bertoin '15, for every $t \geqslant 0$, the family of the sizes of cells which are present in the system at time t is cube-summable, and can therefore be ranked in non-increasing order. This yields a random variable with values in $\ell_3^{\downarrow}$ which we denote by $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots)$.

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

$\rightsquigarrow$ $\mathbf{X}$ fulfills the branching property, and is self-similar with index $-1/2$

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

$\rightsquigarrow$ $\mathbf{X}$ fulfills the branching property, and is self-similar with index $-1/2$, in the sense that for every $c > 0$, the rescaled process $(c\mathbf{X}(\mathbf{c}^{-1/2}\mathbf{t}), t \geqslant 0)$ has the same law as $\mathbf{X}$ started from the sequence $(c, 0, 0, \ldots)$.

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

⇝ $\mathbf{X}$ fulfills the branching property, and is self-similar with index $-1/2$, in the sense that for every $c > 0$, the rescaled process $(c\mathbf{X}(\mathbf{c}^{-1/2}\mathbf{t}), t \geqslant 0)$ has the same law as $\mathbf{X}$ started from the sequence $(c, 0, 0, \dots)$.

⇝ The dislocations occurring in $\mathbf{X}$ are binary, i.e. they correspond to replacing some mass m in the system by two smaller masses m_1 and m_2 with $m_1 + m_2 = m$.

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

⇝ $\mathbf{X}$ fulfills the branching property, and is self-similar with index $-1/2$, in the sense that for every $c > 0$, the rescaled process $(c\mathbf{X}(\mathbf{c}^{-1/2}\mathbf{t}), t \geqslant 0)$ has the same law as $\mathbf{X}$ started from the sequence $(c, 0, 0, \ldots)$.

⇝ The dislocations occurring in $\mathbf{X}$ are binary, i.e. they correspond to replacing some mass m in the system by two smaller masses m_1 and m_2 with $m_1 + m_2 = m$. Informally, in $\mathbf{X}$, each mass $m > 0$ splits into a pair of smaller masses $(xm, (1-x)m)$ at rate $m^{-1/2}\nu(dx)$, where

$$\nu(dx) = (x(1-x))^{-5/2}dx.$$

Description of the growth-fragmentation

We can think of $\mathbf{X}$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:

↝ $\mathbf{X}$ fulfills the branching property, and is self-similar with index $-1/2$, in the sense that for every $c > 0$, the rescaled process $(c\mathbf{X}(\mathbf{c^{-1/2}t}), t \geqslant 0)$ has the same law as $\mathbf{X}$ started from the sequence $(c, 0, 0, \ldots)$.

↝ The dislocations occurring in $\mathbf{X}$ are binary, i.e. they correspond to replacing some mass m in the system by two smaller masses m_1 and m_2 with $m_1 + m_2 = m$. Informally, in $\mathbf{X}$, each mass $m > 0$ splits into a pair of smaller masses $(xm, (1-x)m)$ at rate $m^{-1/2}\nu(dx)$, where

$$\nu(dx) = (x(1-x))^{-5/2}dx.$$

↝ We have $\int(1-x)^2\nu(dx) < \infty$, but $\int(1-x)\nu(dx) = \infty$ which underlines the necessity of compensating the dislocations.

Cycles and growth-fragmentations

Recall that $\mathbf{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Cycles and growth-fragmentations

Recall that $\mathbf{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \dots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Theorem (Bertoin, Curien, K. '15).

We have

$$\left(\frac{1}{p} \cdot \mathbf{L}^{(p)}(r\sqrt{p}); r \geqslant 0\right) \quad \xrightarrow[p\to\infty]{(d)} \quad \left(\mathbf{X}\left(\frac{3}{2\sqrt{\pi}} \cdot r\right); r \geqslant 0\right),$$

Cycles and growth-fragmentations

Recall that $\mathbf{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Theorem (Bertoin, Curien, K. '15)**.**

We have

$$\left(\frac{1}{p} \cdot \mathbf{L}^{(p)}(r\sqrt{p}); r \geqslant 0\right) \quad \xrightarrow[p\to\infty]{(d)} \quad \left(\mathbf{X}\left(\frac{3}{2\sqrt{\pi}} \cdot r\right); r \geqslant 0\right),$$

where $\mathbf{X} = (\mathbf{X}(t); t \geqslant 0)$ is a self-similar growth-fragmentation process with index $-1/2$ associated with ξ.

Cycles and growth-fragmentations

Recall that $\mathbf{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Theorem (Bertoin, Curien, K. '15).

We have

$$\left(\frac{1}{p} \cdot \mathbf{L}^{(p)}(r\sqrt{p}); r \geqslant 0\right) \quad \xrightarrow[p\to\infty]{(d)} \quad \left(\mathbf{X}\left(\frac{3}{2\sqrt{\pi}} \cdot r\right); r \geqslant 0\right),$$

where $\mathbf{X} = (\mathbf{X}(t); t \geqslant 0)$ is a self-similar growth-fragmentation process with index $-1/2$ associated with ξ. The convergence holds in distribution in the space of càdlàg process taking values in $\ell_3^{\downarrow}$ equipped with the Skorokhod topology.

Cycles and growth-fragmentations

Recall that $\mathbf{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \dots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Theorem (Bertoin, Curien, K. '15).

We have

$$\left(\frac{1}{p} \cdot \mathbf{L}^{(p)}(r\sqrt{p}); r \geqslant 0\right) \quad \xrightarrow[p\to\infty]{(d)} \quad \left(\mathbf{X}\left(\frac{3}{2\sqrt{\pi}} \cdot r\right); r \geqslant 0\right),$$

where $\mathbf{X} = (\mathbf{X}(t); t \geqslant 0)$ is a self-similar growth-fragmentation process with index $-1/2$ associated with ξ. The convergence holds in distribution in the space of càdlàg process taking values in $\ell_3^{\downarrow}$ equipped with the Skorokhod topology.

Recall that

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} \left(x^q - 1 + q(1-x)\right)\left(x(1-x)\right)^{-5/2} dx.$$

Cycles and growth-fragmentations

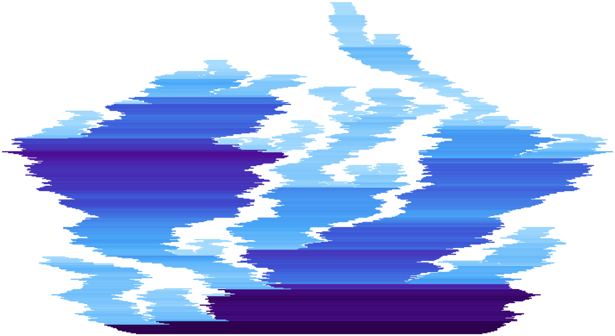


Figure: An artistic representation of the cycle lengths of a Boltzmann triangulation with a large boundary obtained by slicing it at all heights: horizontal line segments correspond to the lengths of the cycles of the ball of radius r of the triangulation as r increases. Here the longest cycles are the darkest ones.