Problem Set I

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1. (IMO, 2000)

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\le 1 \ \ (abc=1,\ a,b,c>0)$$

2. (IMO Short-listed, 1998)

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\ge\frac{3}{4} \ \ (xyz=1,\ x,y,z>0)$$

3. (IMO Short-listed, 1996)

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1 \ \ (abc=1,\ a,b,c>0)$$

4. (IMO, 1995)

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\ge\frac{3}{2} \ \ (abc=1,\ a,b,c>0)$$

5. (IMO Short-listed, 1993)

$$abc+bcd+cda+dab\le\frac{1}{27}+\frac{176}{27}abcd \ \ (a+b+c+d=1,\ a,b,c,d>0)$$

6. (IMO Short-listed, 1993)

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}\ge\frac{2}{3} \ \ (a,b,c,d>0)$$

7. (IMO Short-listed, 1990)

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\ge\frac{1}{3} \ \ (ab+bc+cd+da=1,\ a,b,c,d>0)$$

8. (IMO, 1984)

$$0\le xy+yz+zx-2xyz\le\frac{7}{27} \ \ (x+y+z=1,\ x,y,z\ge 0)$$

9. (IMO, 1968)

$$\frac{1}{x_1y_1-{z_1}^2}+\frac{1}{x_2y_2-{z_2}^2}\ge\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2}$$

$$(x_1,x_2>0,y_1,y_2,z_1,z_2\in R,x_1y_1>{z_1}^2,x_2y_2>{z_2}^2)$$

10. (Poland, 1992)

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2) \ \ (a,b,c \in R)$$

11. (KMO Winter Program Test, 2001)

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} \ \ (a,b,c>0)$$

12. (KMO Summer Program Test, 2001)

$$\sqrt{a^4+b^4+c^4}+\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq \sqrt{a^3b+b^3c+c^3a}+\sqrt{ab^3+bc^3+ca^3} \ \ (a,b,c>0)$$

13. (United Kingdom, 1999)

$$7(pq+qr+rp) \leq 2+9pqr \ \ (p+q+r=1, \ p,q,r>0)$$

14. (Romania, 1997)

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab} \geq 1 \geq \frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab} \ \ (a,b,c>0)$$

15. (Japan, 1997)

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5} \ \ (a,b,c>0)$$

16. (Belarus, 1999)

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca} \geq \frac{3}{2} \ \ (a^2+b^2+c^2=3, \ a,b,c>0)$$

17. (Iran, 1996)

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \geq \frac{9}{4} \ \ (a,b,c>0)$$

18. (Hong Kong, 1994)

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9} \ \ (xy+yz+zx=1, \ x,y,z>0)$$

19. (Yugoslavia, 1987)

$$\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b) \geq a\sqrt{b}+b\sqrt{a} \ \ (a,b>0)$$

20. (Vietnam, 1996)

$$a+b+c+d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

$$(2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab=16, a,b,c,d \geq 0)$$

21. (Belarus, 1997)

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}\ \ (a,b,c>0)$$

22. (Poland, 1996)

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}\ \ \left(a+b+c=1,\ a,b,c\geq-\frac{3}{4}\right)$$

23. (Bulgaria, 1997)

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}\ \ (abc=1,\ a,b,c>0)$$

24.

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq\frac{3\sqrt{2}}{2}\ \ (a,b,c\in R)$$

25.

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}\geq\sqrt{a^2+ac+c^2}\ \ (a,b,c>0)$$

26.

$$\sqrt[3]{xyz}+\frac{|x-y|+|y-z|+|z-x|}{3}\geq\frac{x+y+z}{3}\ \ (x,y,z>0)$$

27.

$$\sqrt[3]{(a+x)(b+y)(c+z)}\geq\sqrt[3]{abc}+\sqrt[3]{xyz}\ \ (a,b,c,x,y,z>0)$$

28.

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq1\ \ (x,y,z>0)$$

29.

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}}\geq\sqrt{\frac{3}{2}}\ \ (x+y+z=1,\ x,y,z>0)$$

30.

$$-\frac{1}{2}\leq\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\leq\frac{1}{2}\ \ (x,y\in R)$$

31.

$$\frac{x^{2001}}{y+z}+\frac{y^{2001}}{z+x}+\frac{z^{2001}}{x+y}\geq\frac{x^{2000}+y^{2000}+z^{2000}}{2}\ \ (x,y,z>0)$$

32.

$$\frac{1}{\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\geq\frac{1}{3}\ \ (a,b,c>0)$$

33.

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geq\left(\frac{3}{a+b+c}-1\right)^3\ \ \left(0<a,b,c<\frac{1}{2}\right)$$

34.

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\geq\frac{100}{3}\ \ (a+b+c=1,\ a,b,c>0)$$

35.

$$(a-1)(b-1)(c-1)\geq 8\ \ \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1,\ a,b,c>0\right)$$

36.

$$\frac{9}{4(x+y+z)}\geq\frac{x}{(x+y)(x+z)}+\frac{y}{(y+z)(y+x)}+\frac{z}{(z+x)(z+y)}\ \ (x,y,z>0)$$