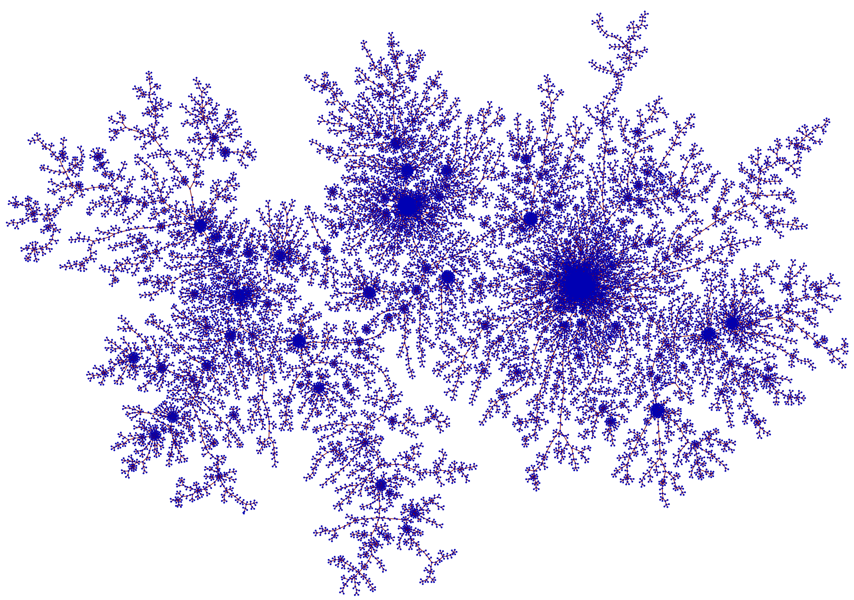
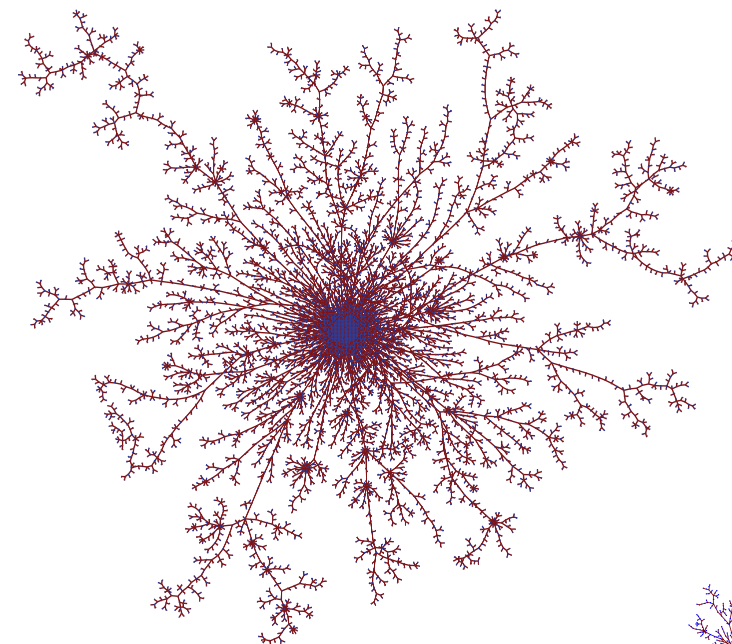
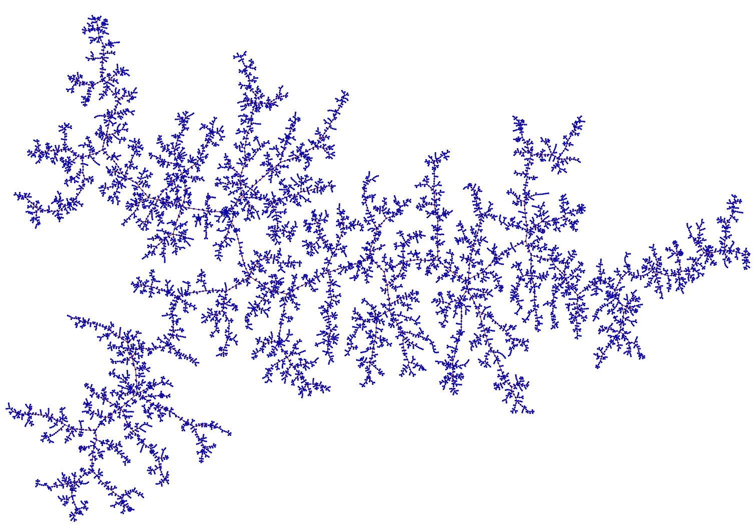
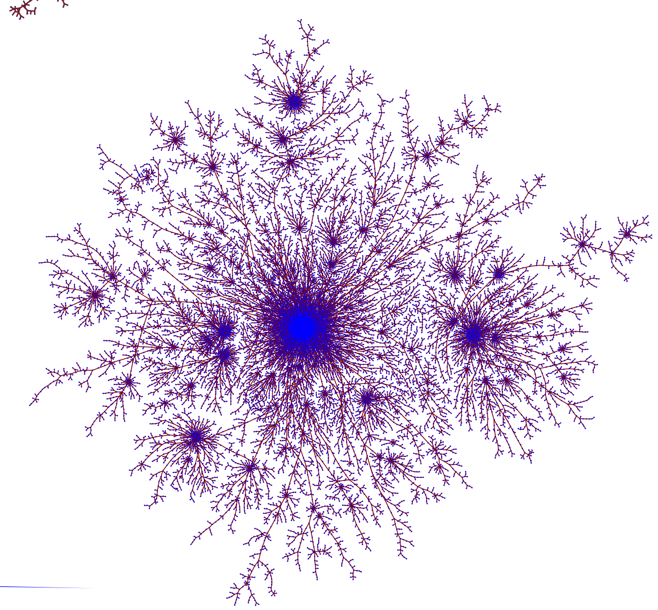
# *Limits of random trees*









Igor Kortchemski
CNRS & ENS Paris

Oxford Probability Workshop : Random Discrete Structures

Today: motivation and an overview of some of the main results in the field.

# Context

Understand the geometry and the structure of large random trees by studying their limits.

# *Motivation for studying limits*

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by $X$ and passes to the limit, $X_n$ satisfies "approximately" $\mathcal{P}$ for $n$ large.

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by $X$ and passes to the limit, $X_n$ satisfies "approximately" $\mathcal{P}$ for $n$ large.
- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards $X$, then $X_n$ and $Y_n$ share approximately the same properties for $n$ large.

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$$X_n \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} X \qquad \text{implies} \qquad G(X_n) \overset{(d)}{\underset{n\to\infty}{\longrightarrow}} G(X)$$

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# Outline

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# Stack triangulations (Albenque, Marckert)

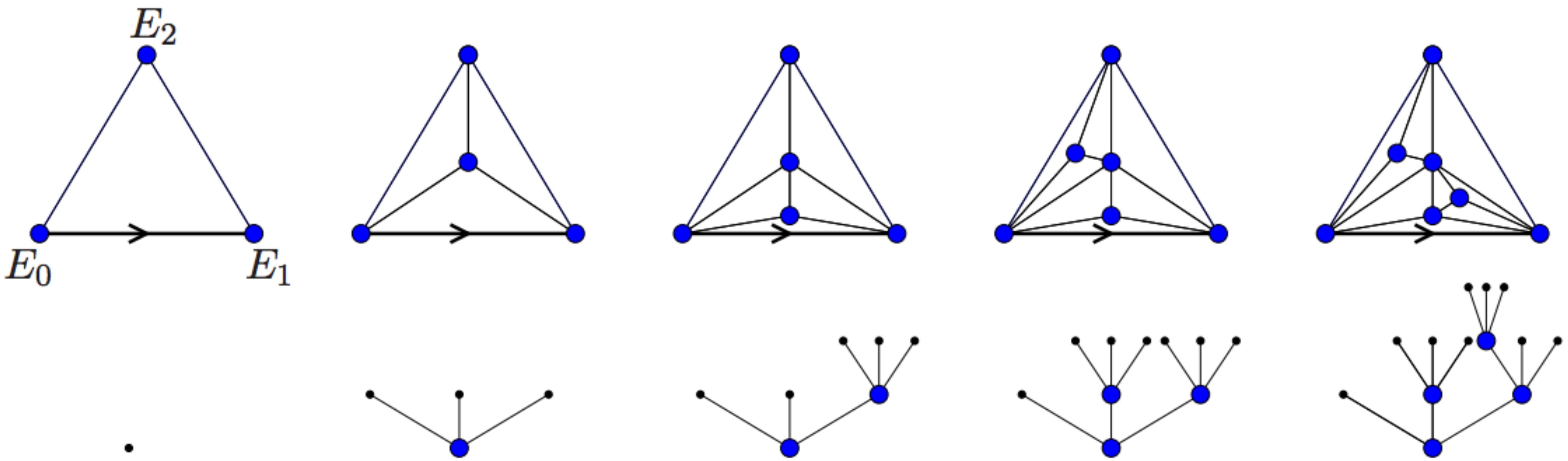



Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

# Maps (Schaeffer)

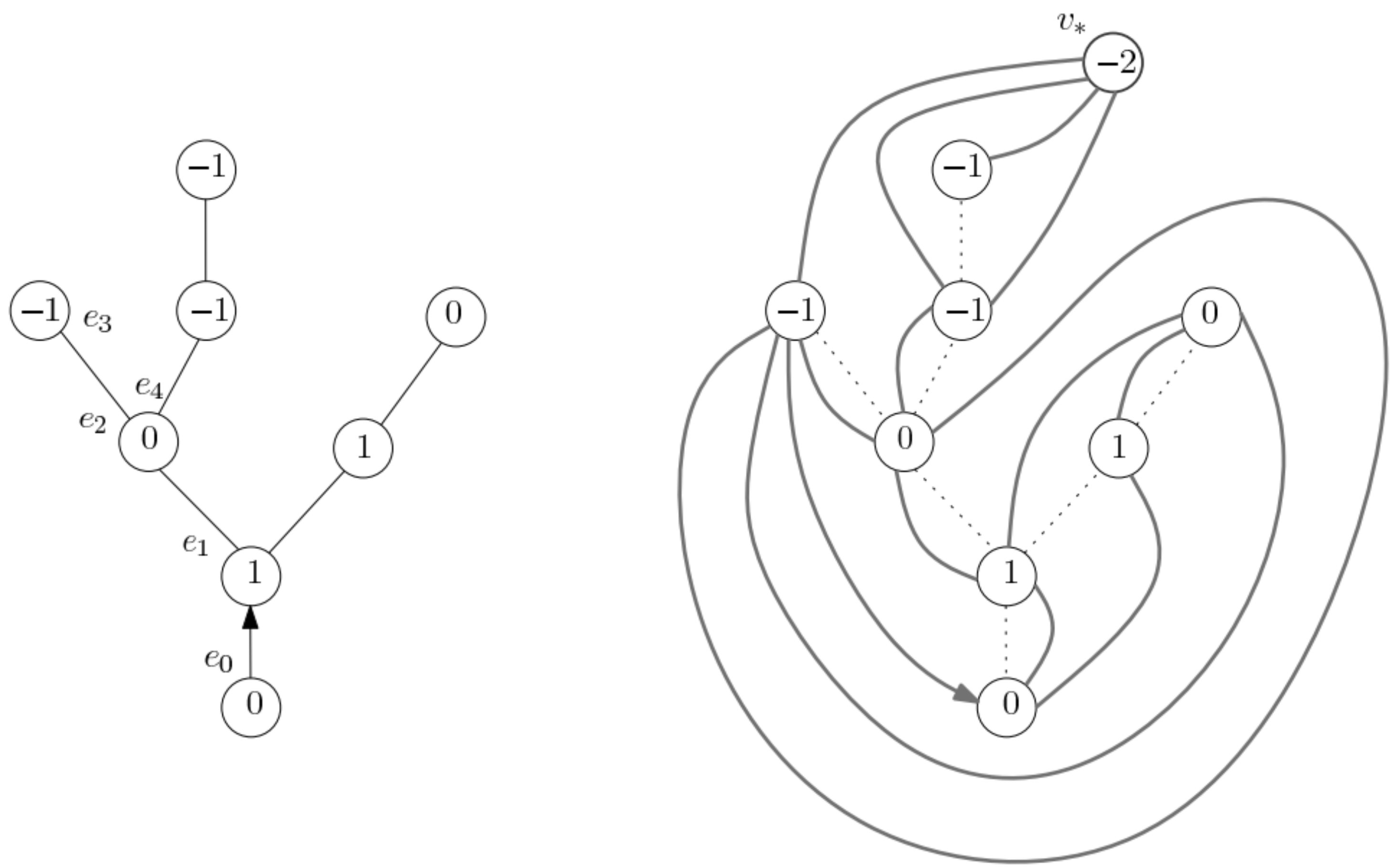
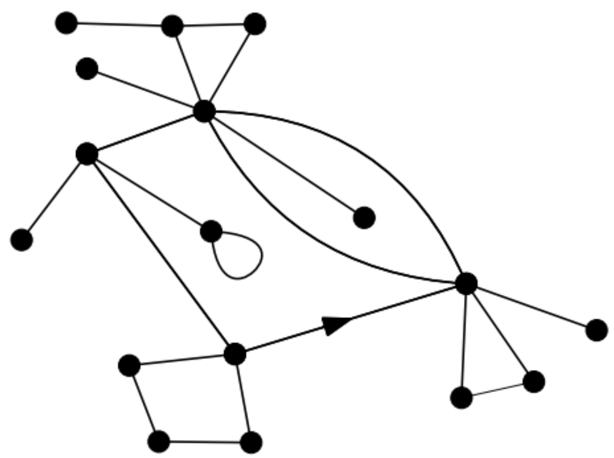
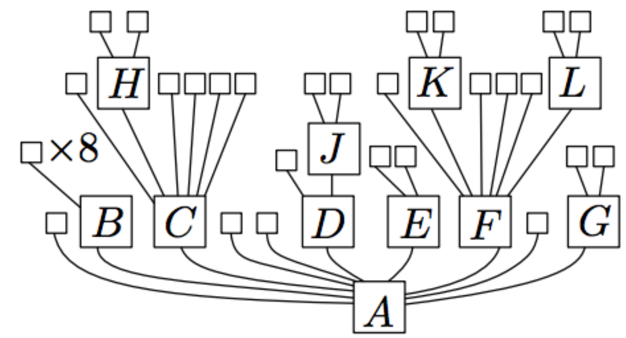



FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, $e_3$ is the successor of $e_0$, $e_2$ the successor of $e_1$, and so on.

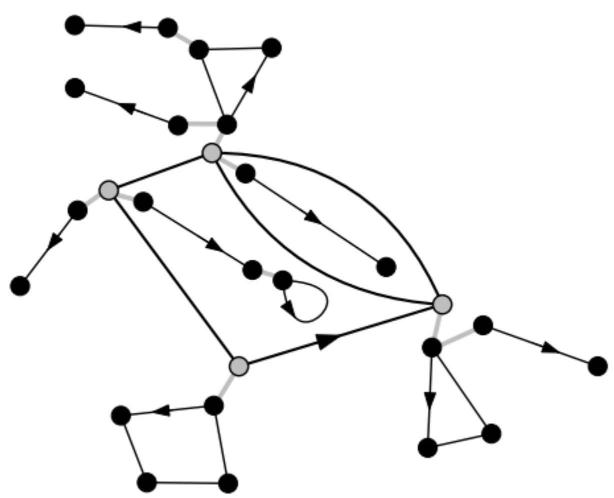
# Maps (Addario-Berry)



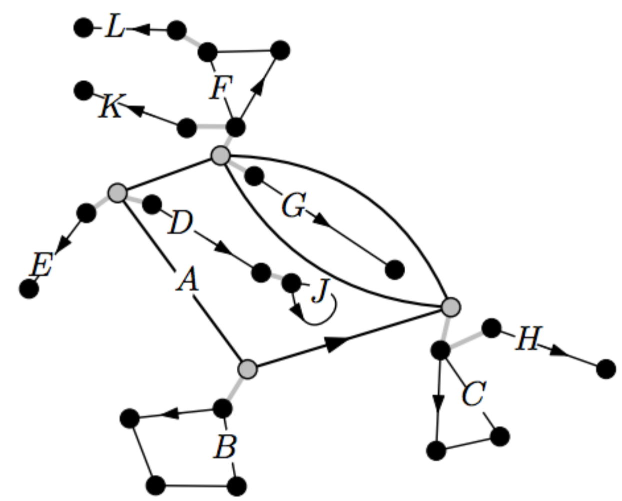
(A) A map M.




(B) The tree $T_M$. Tiny squares represent trivial blocks.

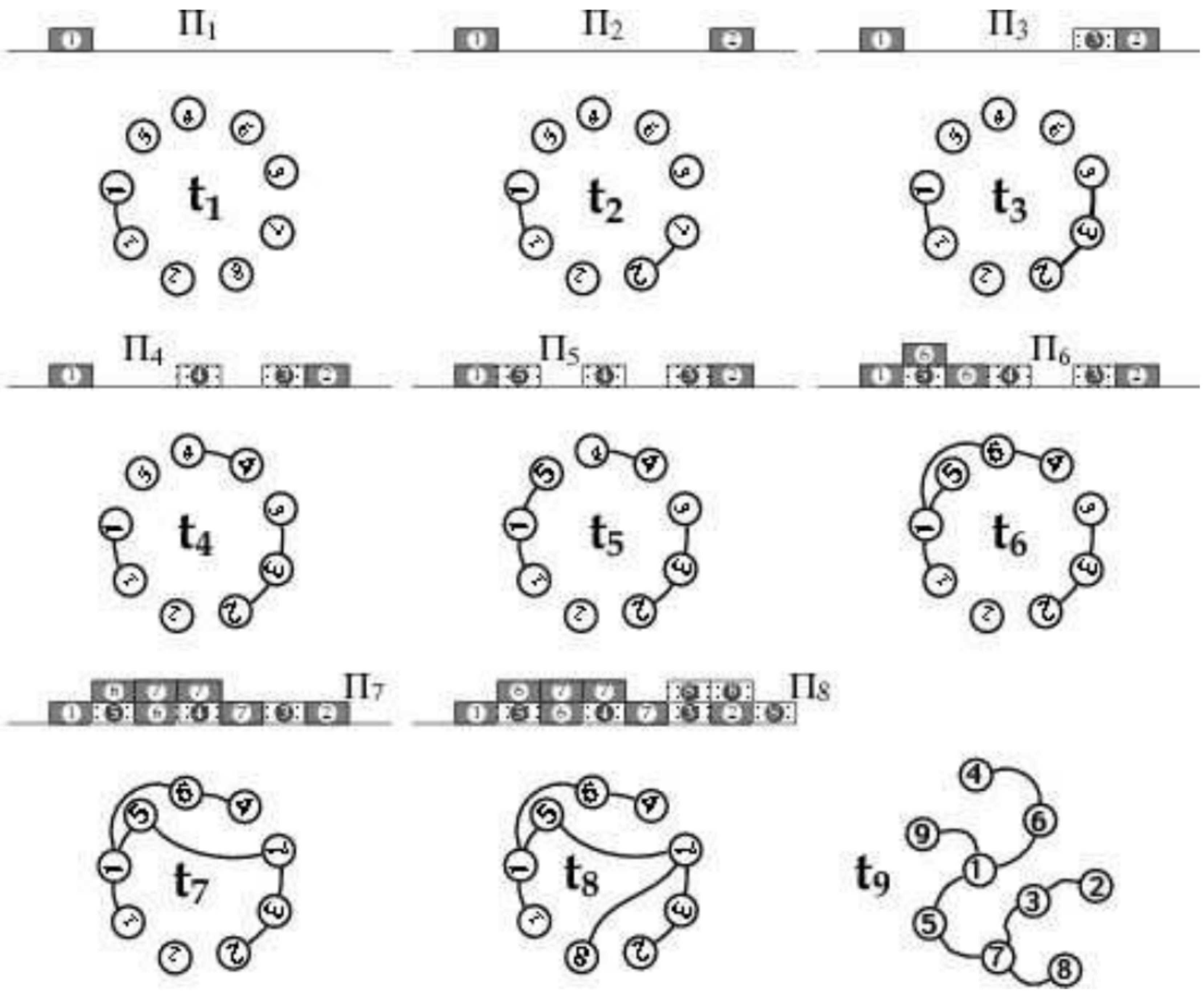


(C) The decomposition of M into blocks. Blocks are joined by grey lines according to the tree structure. Root edges of blocks are shown with arrows.




(D) The correspondence between blocks and nodes of $T_M$. Non-trivial blocks receive the alphabetical label (from $A$ through $L$) of the corresponding node.

# Parking functions (Chassaing, Louchard)



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# Plane trees

Figure: Two different plane trees

We consider plane (i.e. rooted ordered) trees.

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A random tree with law $\mathbb{P}_\mu$ is called a $\mu$-Bienaymé tree (or $B_\mu$ tree).

# Size-conditioned Bienaymé trees

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– when $\mu(k) = e^{-1}/k!$ for $k \geqslant 0$, by labelling the vertices of $\mathcal{T}_n$ uniformly at random and forgetting the order among children, one gets a uniform labelled tree (a.k.a. Cayley tree) with $n$ vertices.

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If $c < \rho_\mu$, two Bienaymé trees with offspring distributions $\mu$ and $\mu_c$, defined by

$$\mu_c(k) = \frac{1}{G_\mu(c)} c^k \mu(k), \qquad k \geqslant 0,$$

when conditioned on having $n$ vertices, have the same distribution (Kennedy '75).

What are the limits of large size-conditioned Bienaymé trees?

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What does a large size-conditioned Bienaymé tree look like, near the root?

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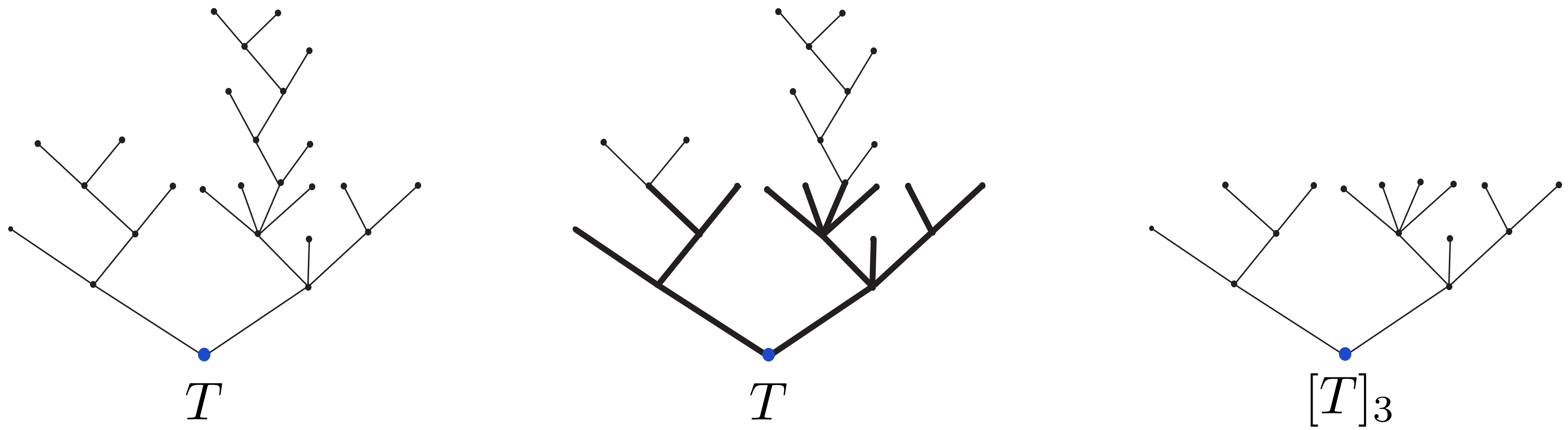
⇝ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of $T$ obtained by keeping the first $k$ generations:



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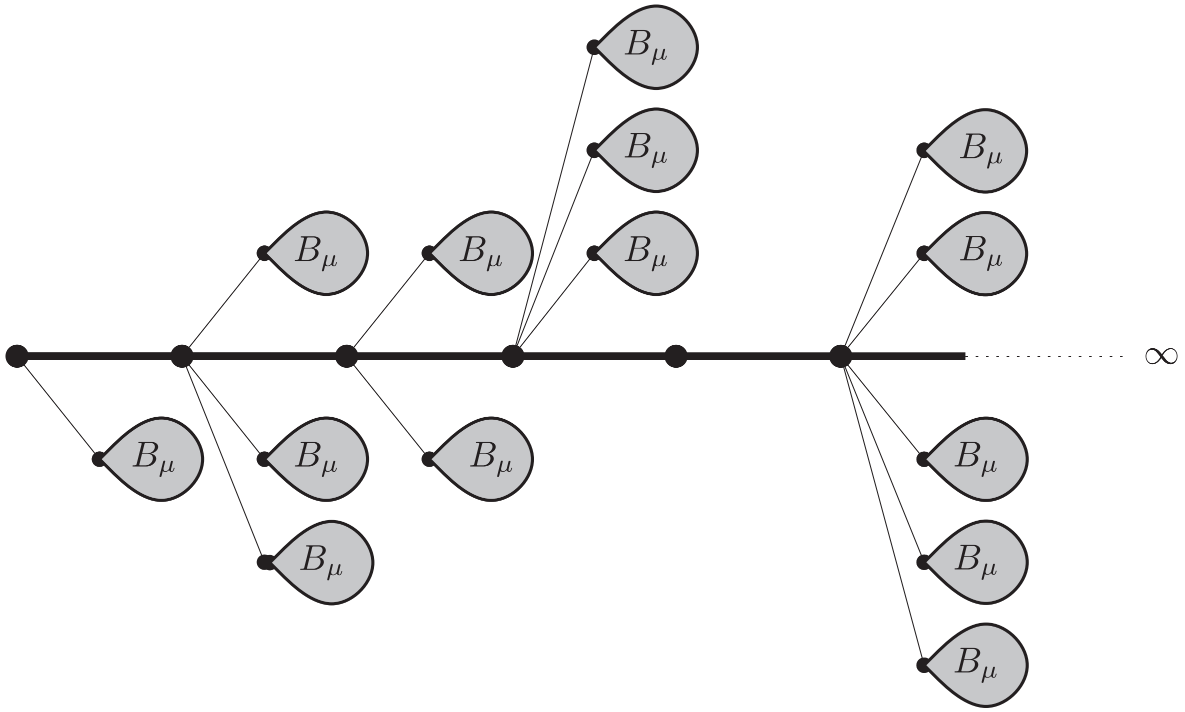

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– length of the left-most path starting from the root?

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$$\mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_\infty$$

*holds in distribution for the local topology, where $\mathcal{T}_\infty$ is the infinite Bienaymé tree conditioned to survive (or Kesten tree).*

$\leadsto$ We will prove this theorem in the mini-course.

# *Local limits: subcritical case*

Let $\mu$ be a **subcritical** offspring distribution and assume that the radius of convergence of $\sum_{i \geqslant 0} \mu(i) z^i$ is $1$.

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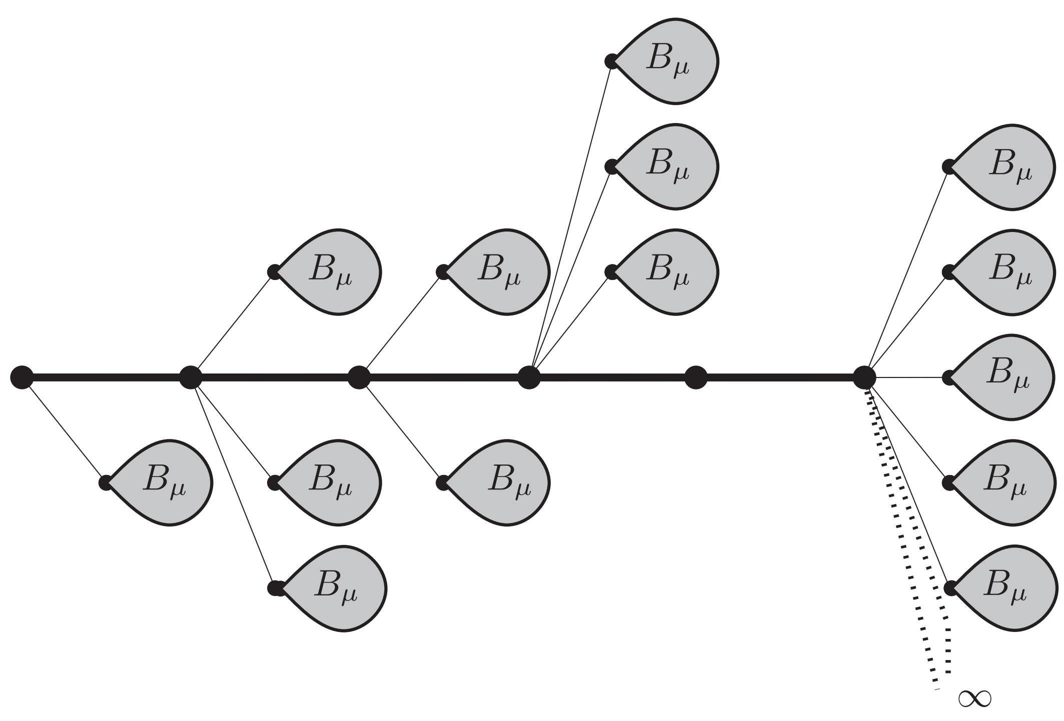

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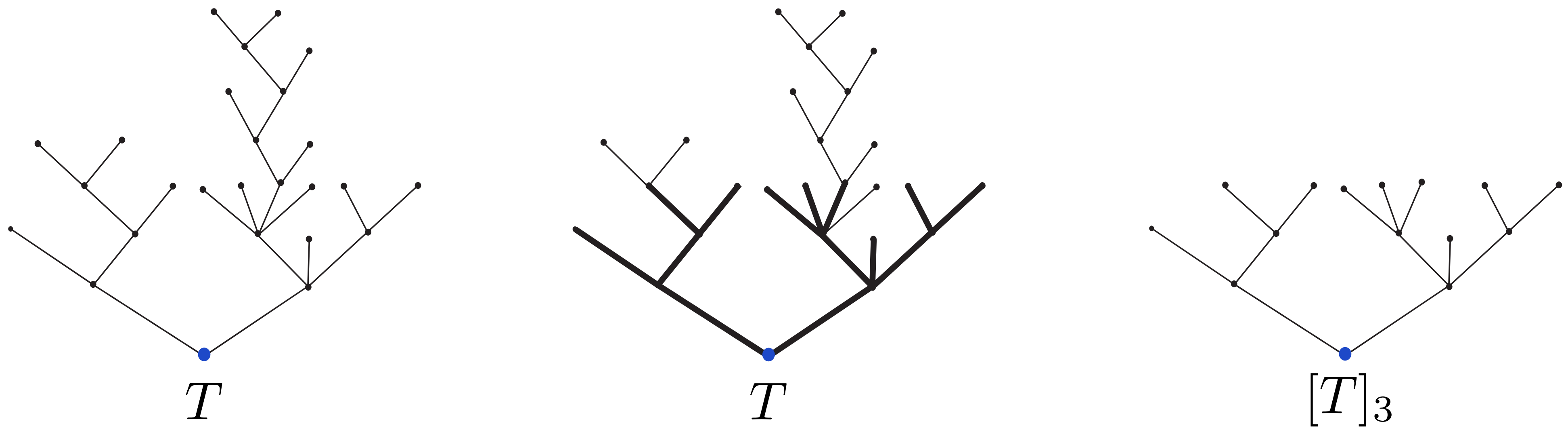
⇝ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of $T$ obtained by keeping *the first* $k$ *children* on the first $k$ generations:



I. Models coded by trees

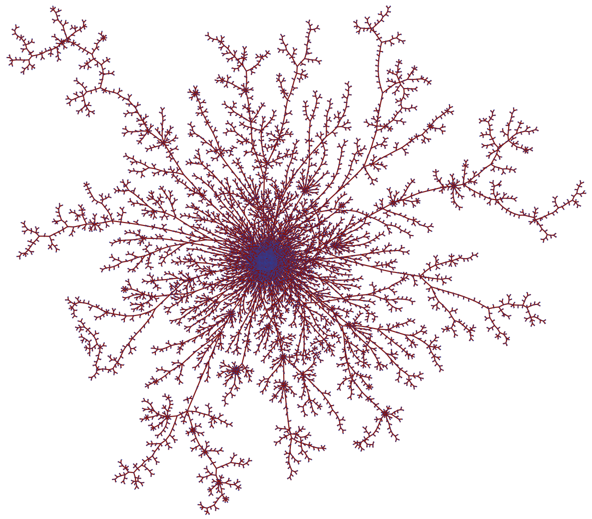
II. Bienaymé trees

III. Local limits of Bienaymé trees

IV. Scaling limits of Bienaymé trees

What does a large Bienaymé tree look like, globally?

# I have simulated and drawn a uniform plane tree with $10000$ vertices. What did I get?



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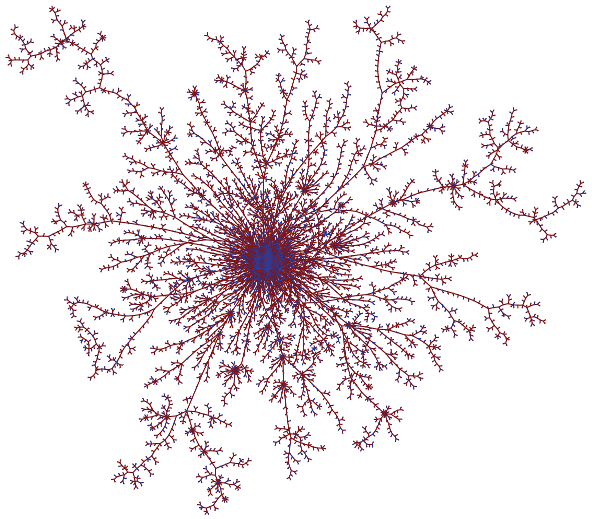

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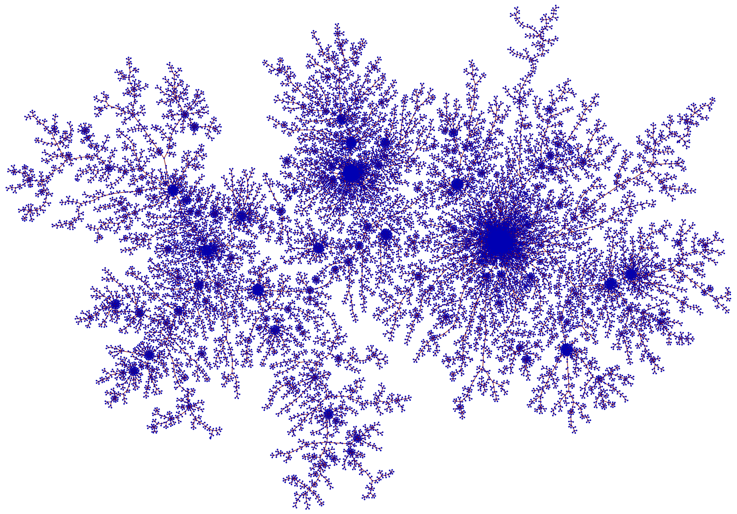

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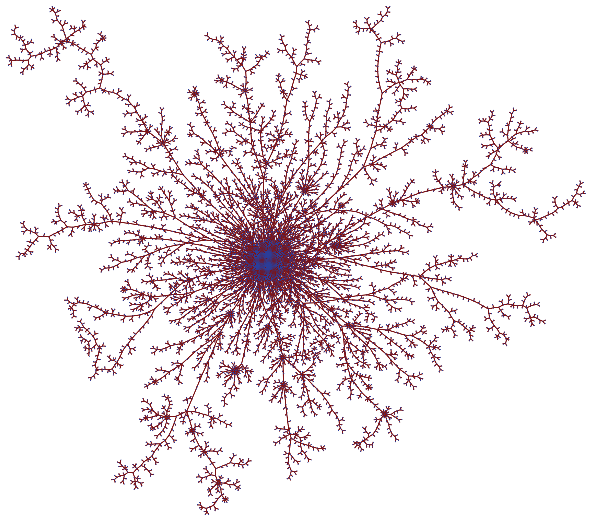

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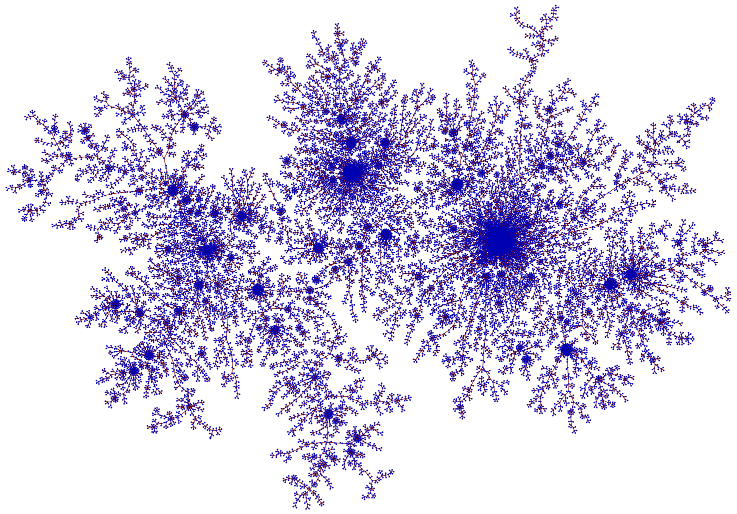

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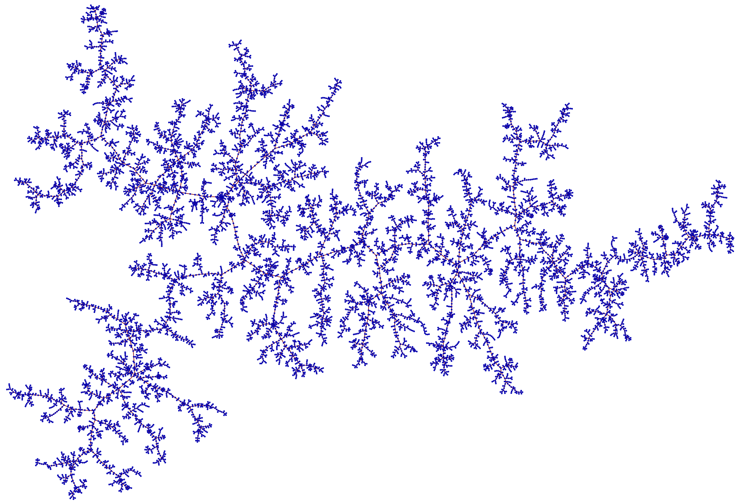

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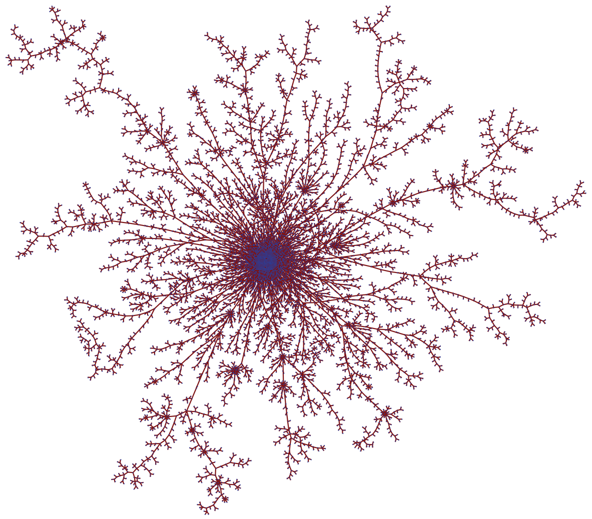

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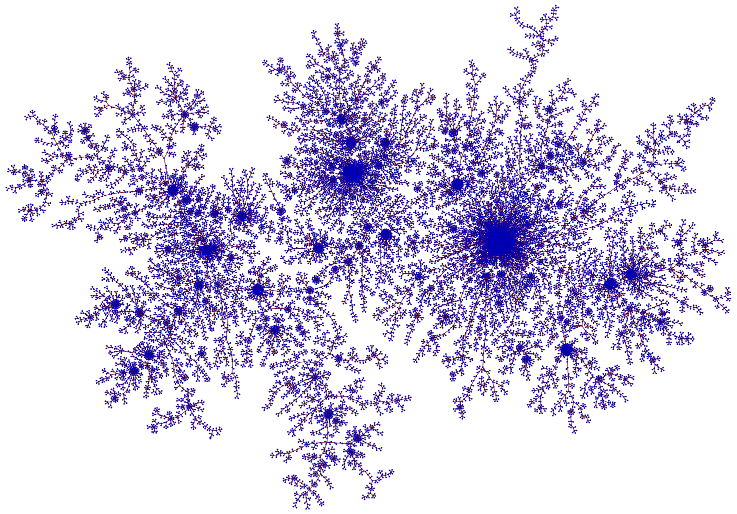

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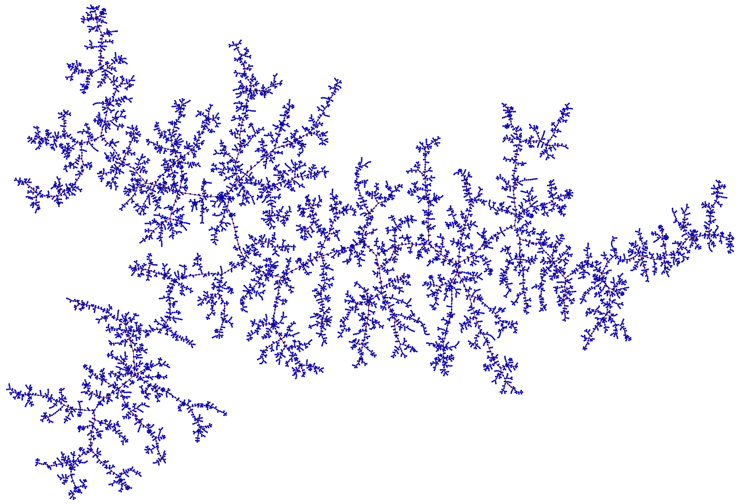

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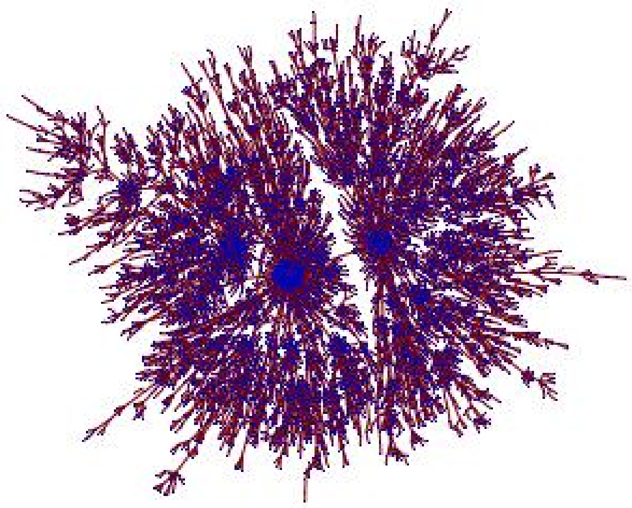

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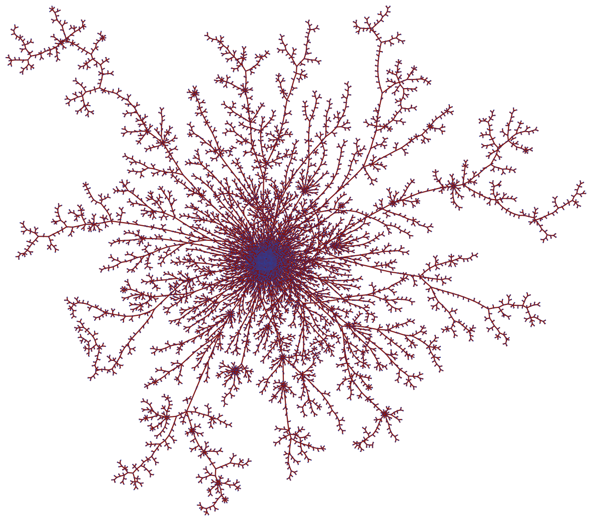

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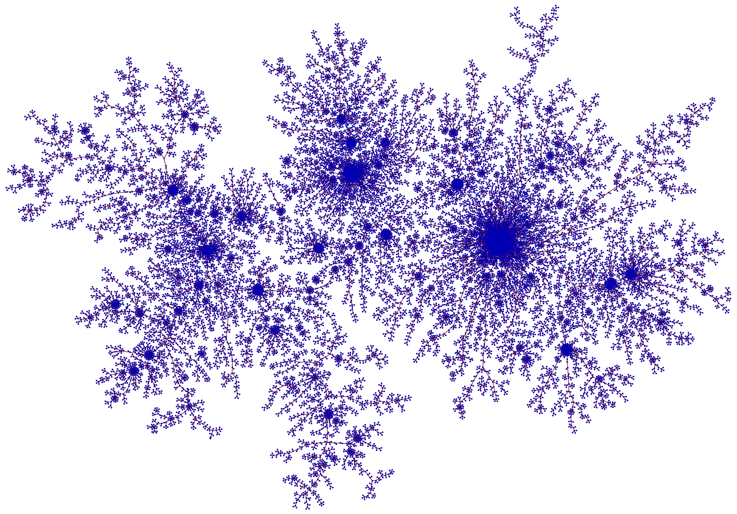

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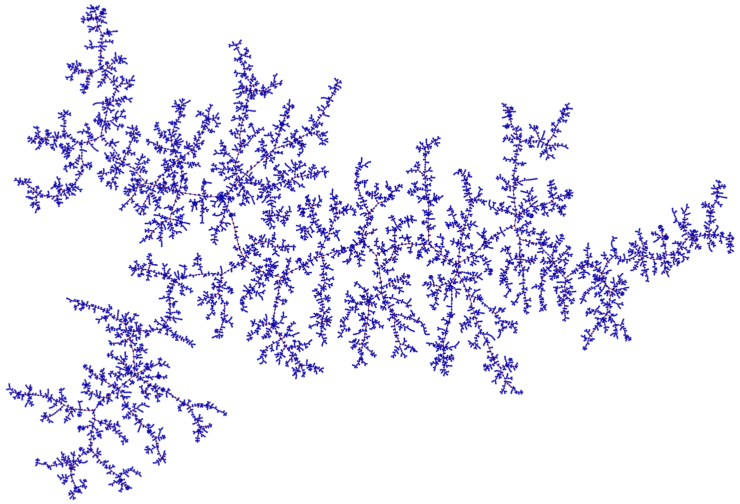

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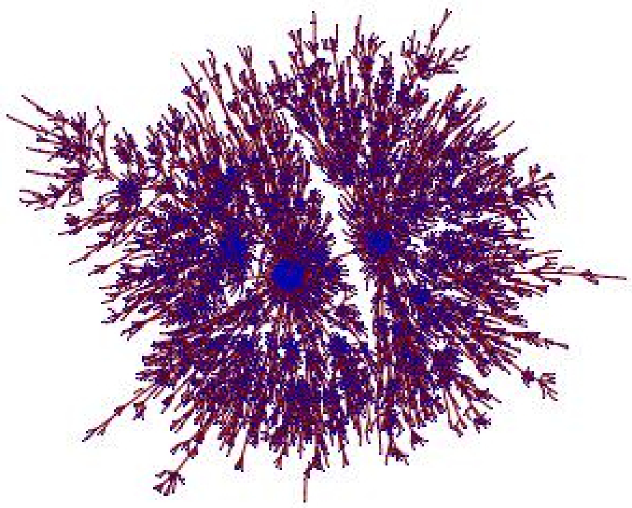

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⇝ wooclap.com ; code **randomtree**.

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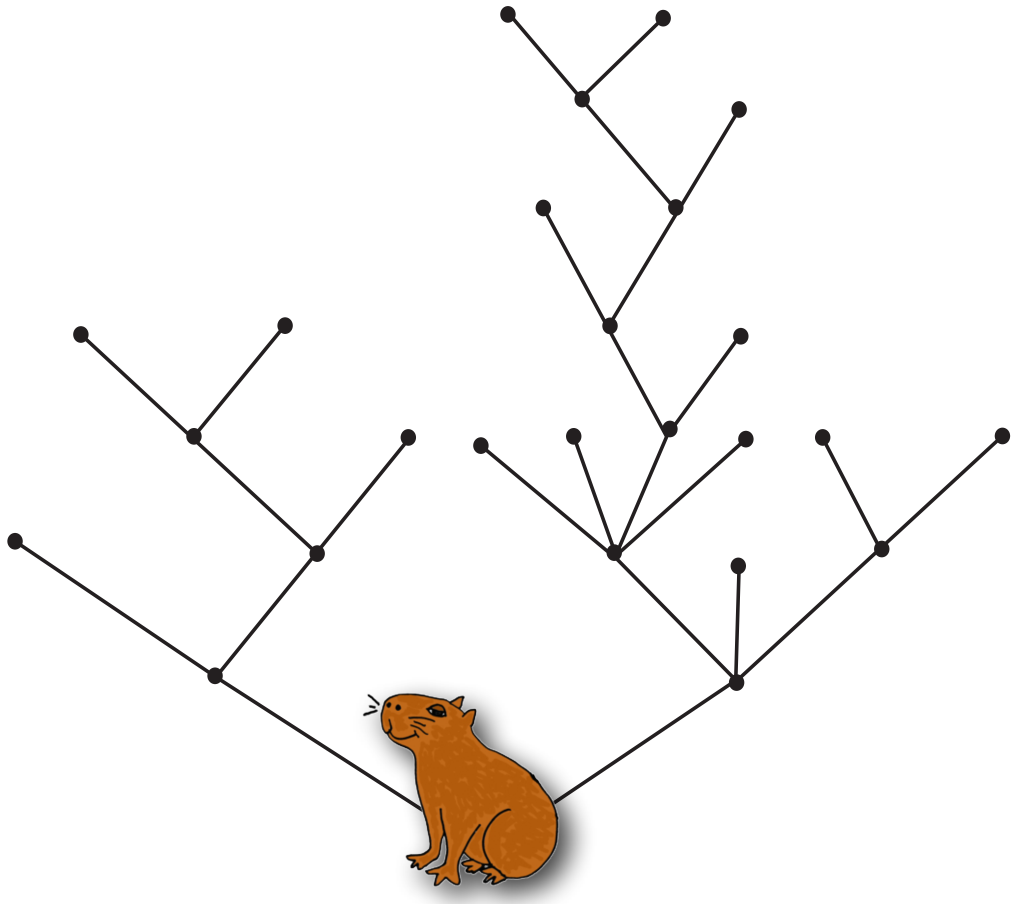
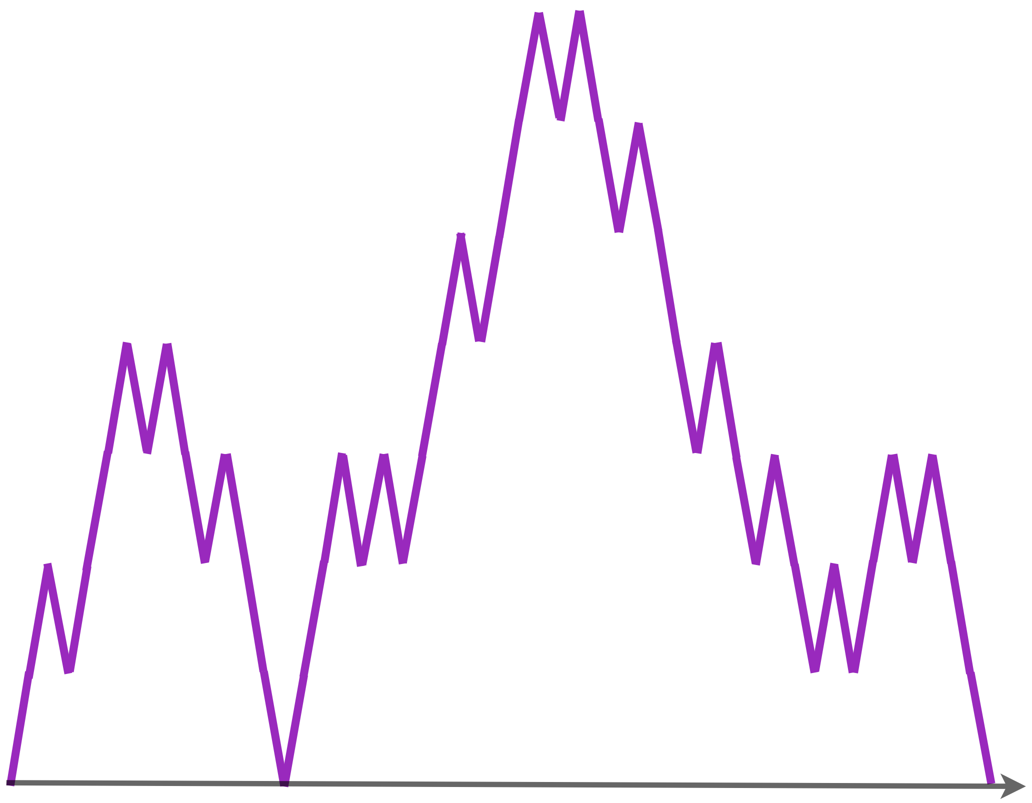
We shall code plane trees by functions.

# Coding trees by functions

# Contour function of a tree

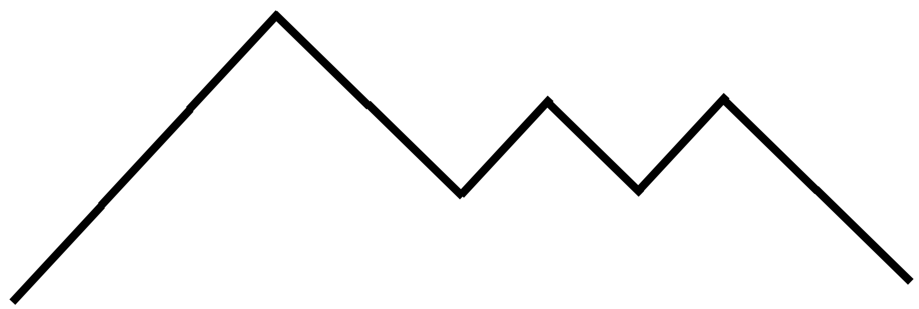
Define the contour function of a tree:





# *Coding trees by contour functions*

Knowing the contour function, it is easy to recover the tree.



# Scaling limits

# *Scaling limits: finite variance*

Let $\mu$ be an offspring distribution with **finite** positive variance such that $\sum_{i \geqslant 0} i\mu(i) = 1$. Let $\mathcal{T}_n$ be a Bienaymé tree conditioned on having $n$ vertices.

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## Theorem (Aldous '93)

*Let $\sigma^2$ be the variance of $\mu$. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:*

$$\left( \frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n) \right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n\to\infty]{(d)}$$

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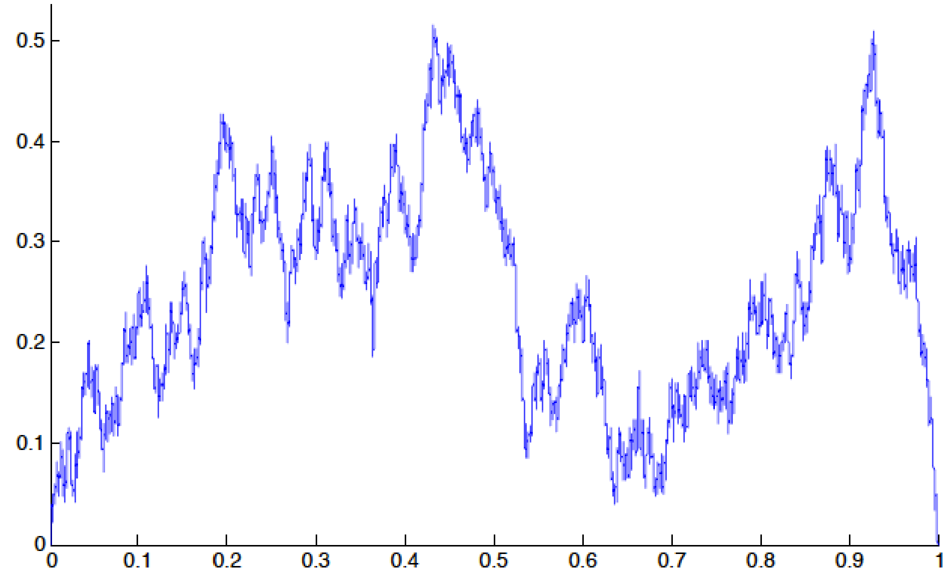

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$\rightsquigarrow$ conditioned Donsker's invariance principle.

# Do the discrete trees converge to a continuous tree?

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*Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!*

# *The Hausdorff distance*

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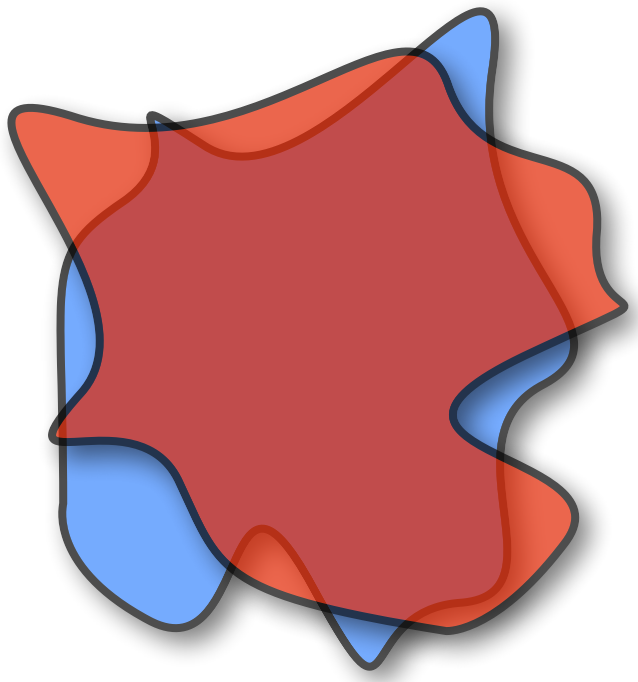

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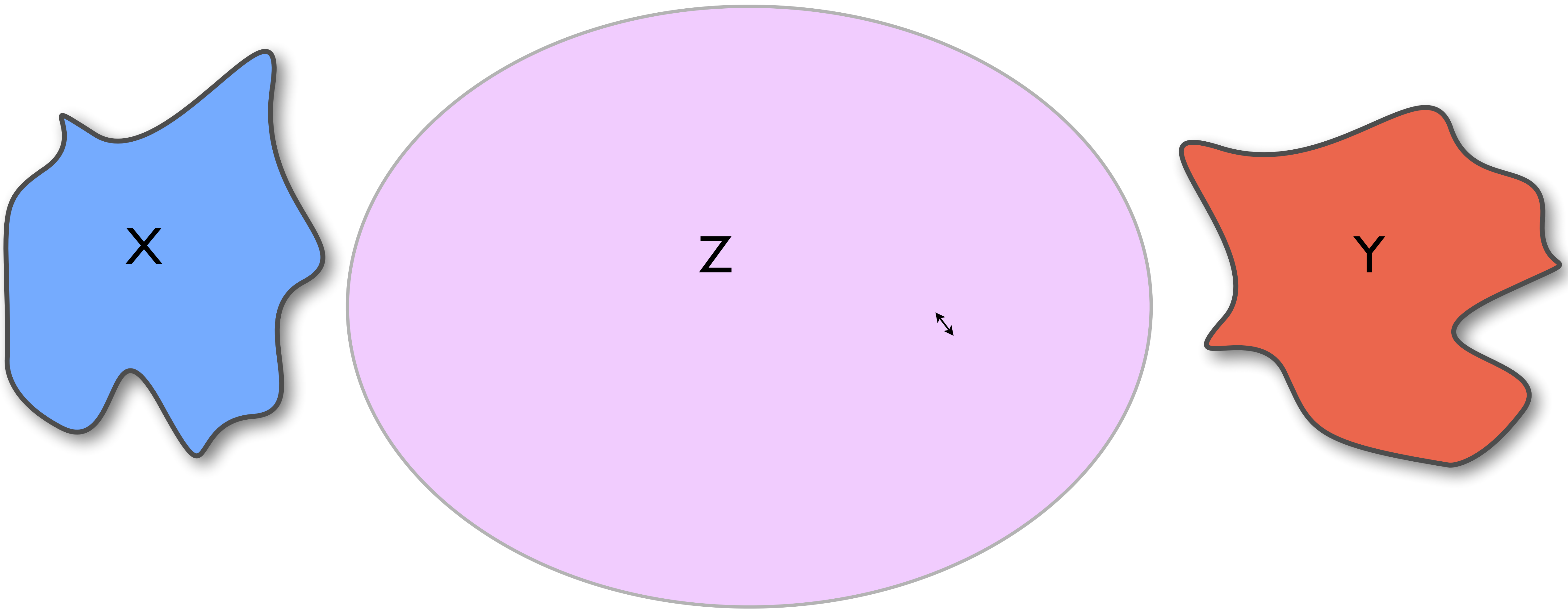


The Gromov–Hausdorff distance between $X$ and $Y$ is the smallest Hausdorff distance between all possible isometric embeddings of $X$ and $Y$ in a *same* metric space $Z$.

# *The Brownian tree*

⇝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

$$\frac{\sigma}{2\sqrt{n}} \cdot \mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_{\mathbb{e}},$$

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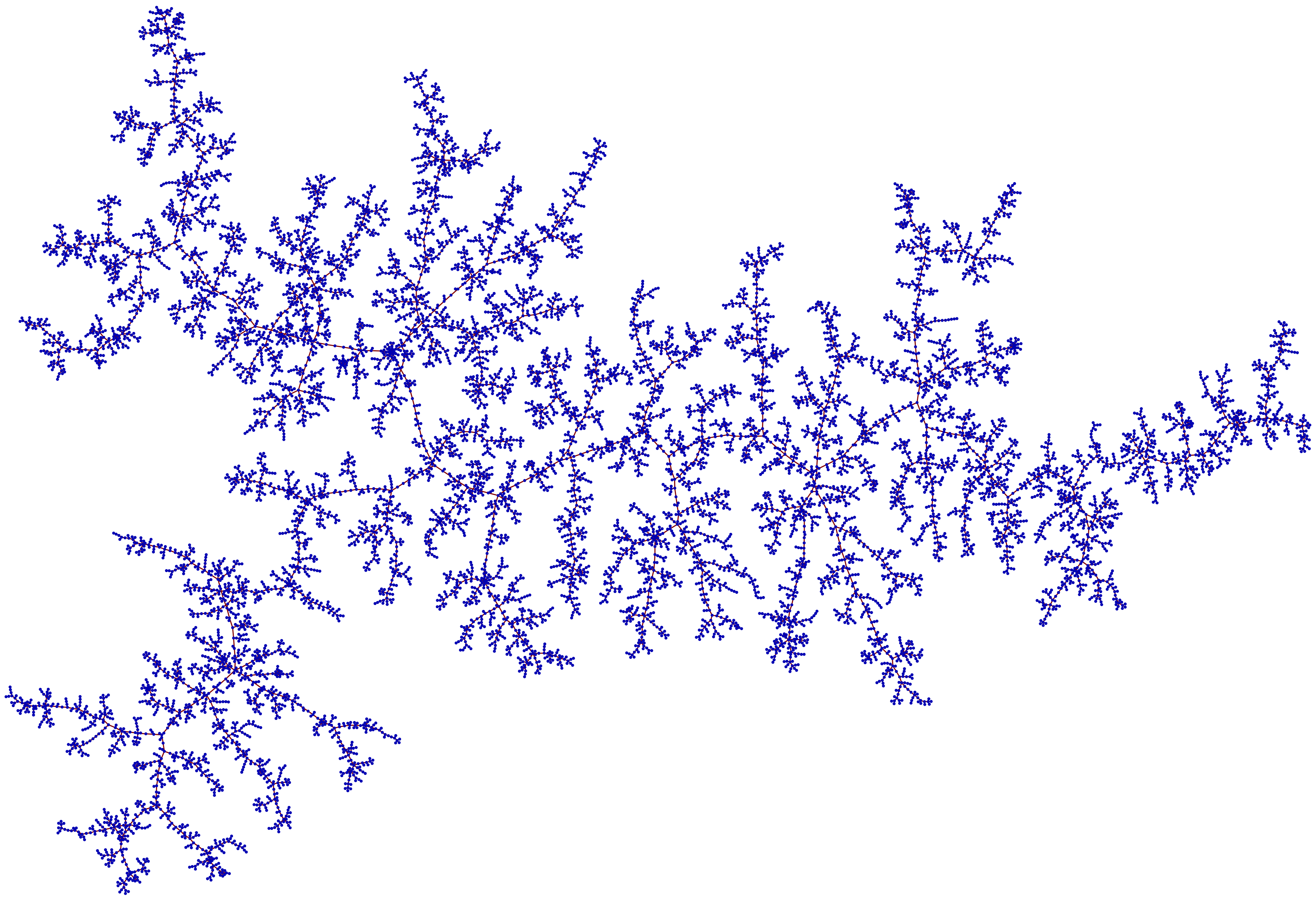
The metric space $\mathcal{T}_{\mathbb{e}}$ is called the *Brownian continuum random tree (CRT)*, and is coded by a Brownian excursion.



An approximation of a realization of a Brownian CRT

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$\rightsquigarrow$ Scaling limits are described use $\alpha$-stable Lévy processes.

# What about non-critical offspring distributions?

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4. Because we condition on total population size, the distribution of $\mathscr{T}_n$ is unchanged by replacing $\xi$ with another distribution $\chi$ in the same exponential family

$$P(\xi = i) = c\theta^i P(\chi = i), \qquad i \geq 0 \text{ for some } c, \theta.$$

Thus there is no essential loss of generality in considering only critical branching processes.

# *Condensation (subcritical case)*

Let $\mu$ be a **subcritical** offspring distribution such that $\mu(n) \sim c/n^{1+\beta}$ with $\beta > 2$. Let $\mathcal{T}_n$ be a $\mu$-Bienaymé tree conditioned on having $n$ vertices.

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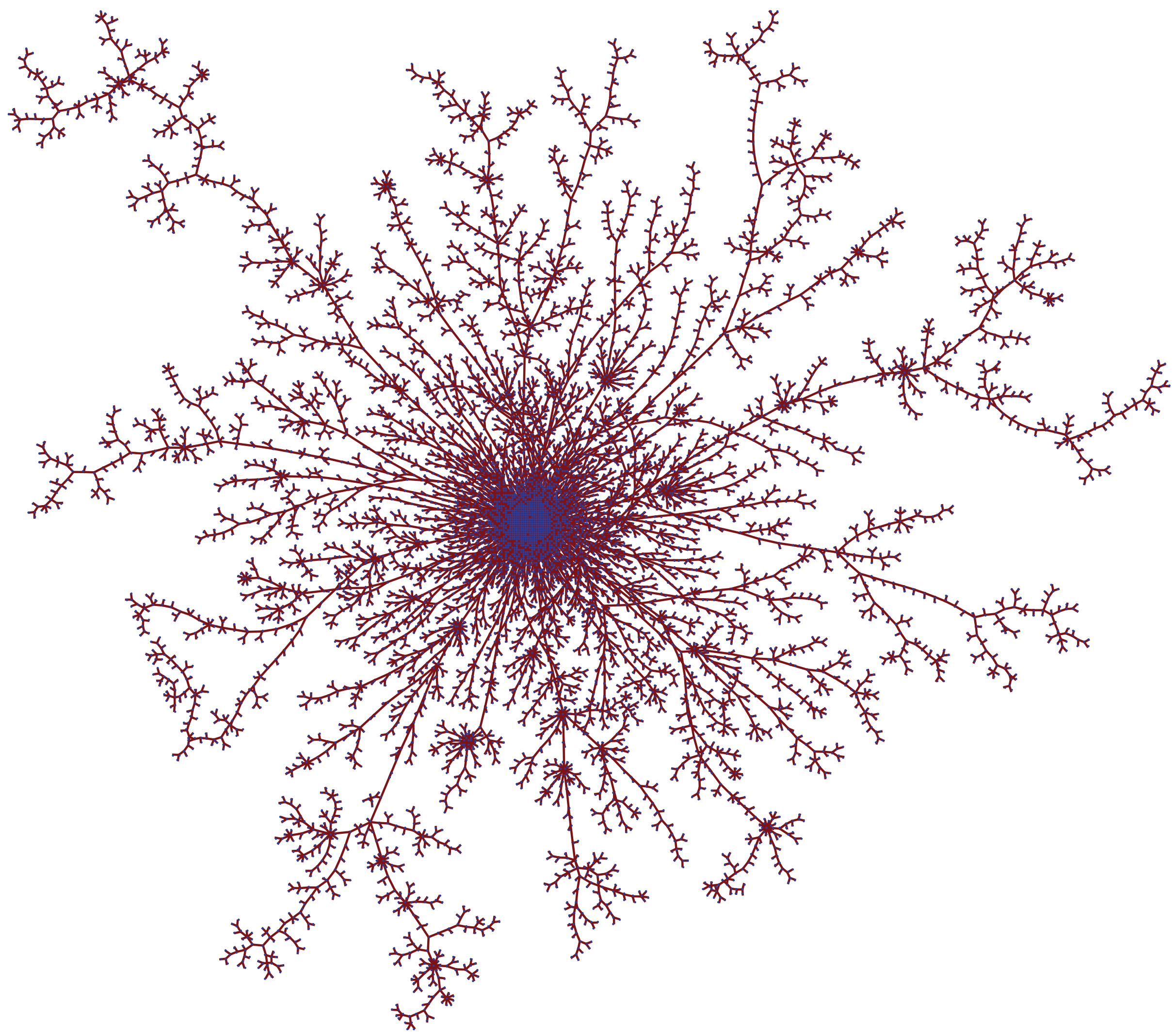

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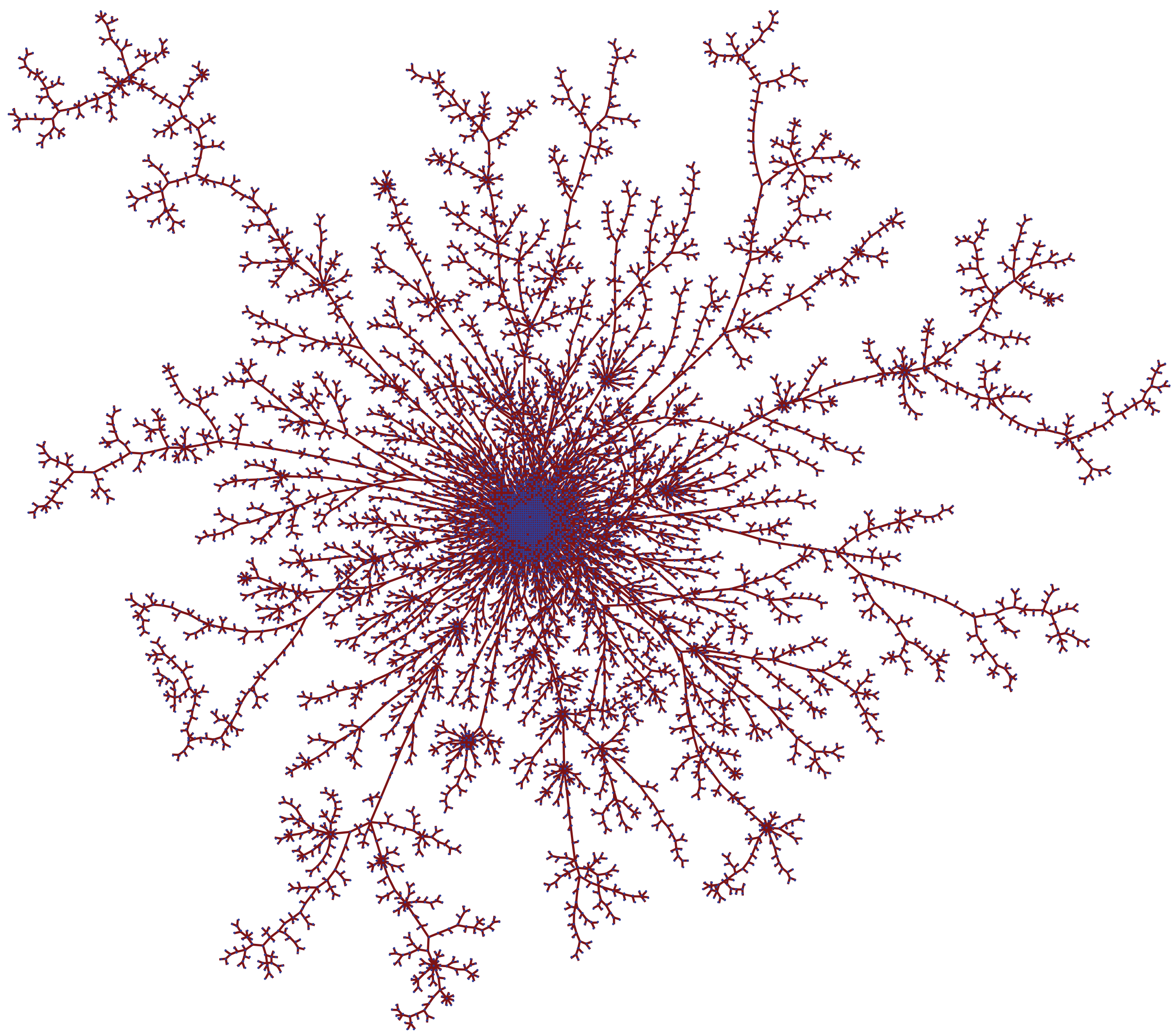

⇝ Situation considered only quite recently by Jonsson & Stefánsson '11!

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*Let $m$ be the mean of $\mu$. Denote by $\Delta(\mathcal{T}_n)$ the maximum degree of $\mathcal{T}_n$. Then*

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$\rightsquigarrow$ What is the order of magnitude of the second largest degree $\Delta^2(\mathcal{T}_n)$?

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## Theorem (Jonsson & Stefánsson '11)

*Let $m$ be the mean of $\mu$. Denote by $\Delta(\mathcal{T}_n)$ the maximum degree of $\mathcal{T}_n$. Then*

$$\frac{\Delta(\mathcal{T}_n)}{n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 1-m.$$

⇝ What is the order of magnitude of the second largest degree $\Delta^2(\mathcal{T}_n)$?

## Theorem (K. '15)

*For every $u > 0$,*

$$\mathbb{P}\left(\frac{\Delta^2(\mathcal{T}_n)}{n^{1/\beta}} \leqslant u\right) \quad \xrightarrow[n\to\infty]{} \quad \exp\left(-\frac{c}{\beta} \cdot \frac{1}{u^{\beta}}\right)$$

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This is not true for any subcritical offspring distribution whose generating function has radius of convergence equal to $1$ (even though there always is a local limit with a finite spine)!

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$\leadsto$ What is the height of $\mathcal{T}_n$?

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## Theorem (K. '15)

*We have*

$$\frac{\textit{Height}(\mathcal{T}_n)}{\ln(n)} \xrightarrow[n\to\infty]{(\mathbb{P})} \ln(1/m).$$

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⇝ Are there non-trivial scaling limits?

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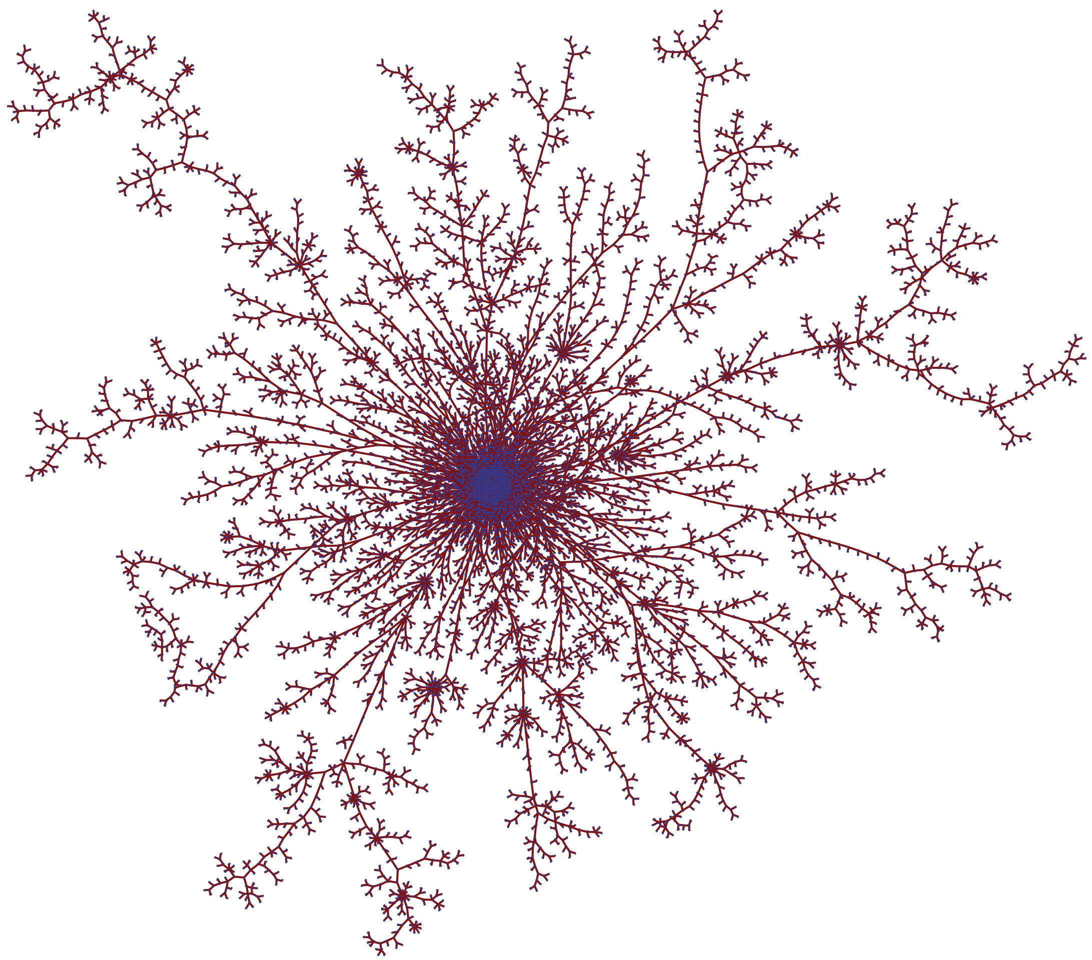
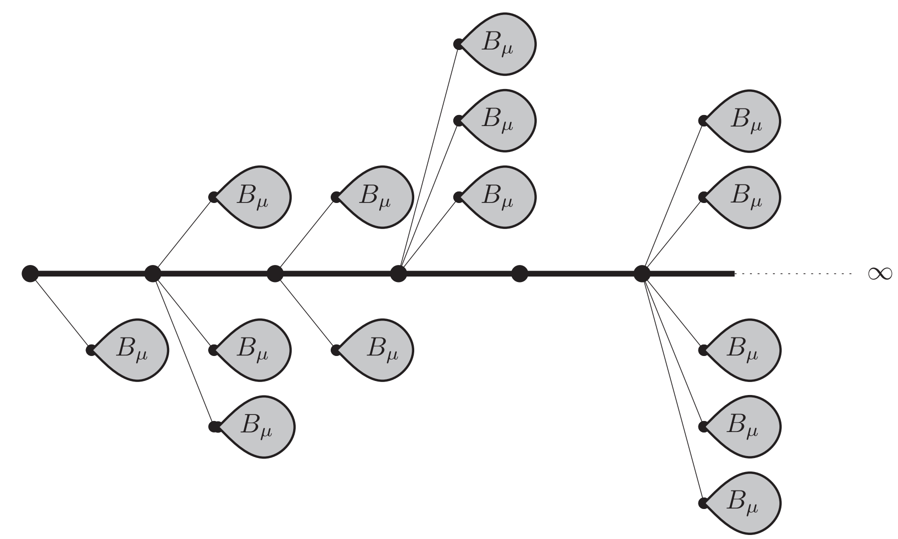
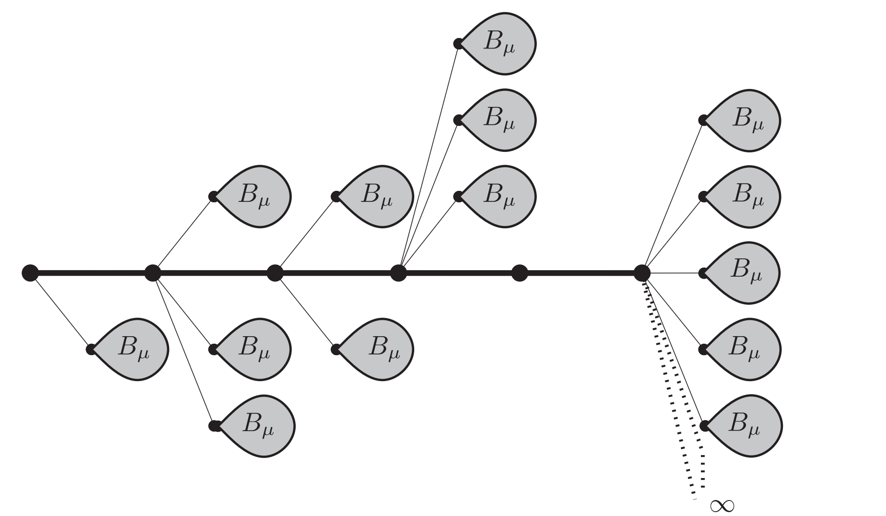
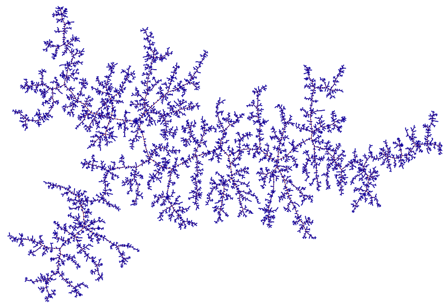
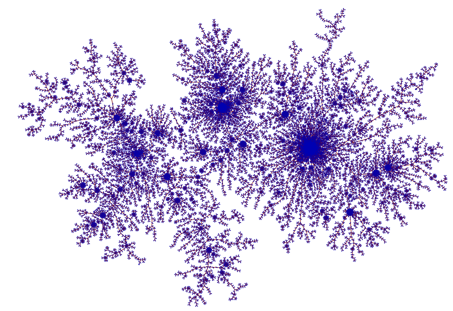
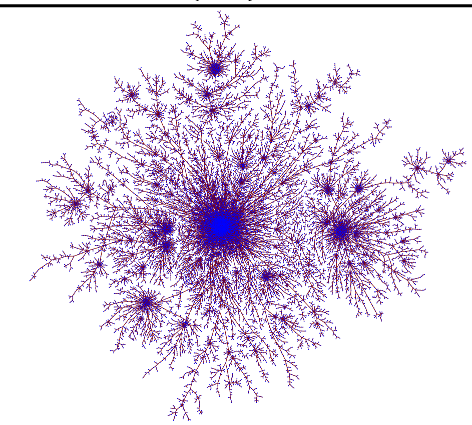
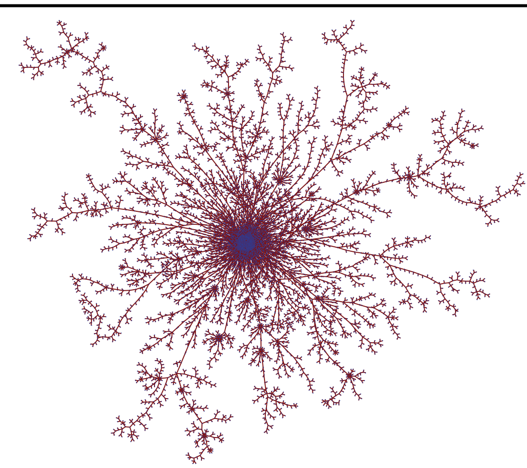


# Recap

# Recap: local limits

$\mathcal{T}_n$ is a $B_\mu$ tree conditioned to have $n$ vertices, and $\rho_\mu$ is the radius of convergence of $G_\mu(z) = \sum \mu(i) z^i$.

| | Local limit of $\mathcal{T}_n$ |
|---|---|
| Critical $\mu$ |   [Kesten '87, Janson '12, Abraham & Delmas '14] |
| Subcritical $\mu$ with $\rho_\mu = 1$ |   [Jonsson & Stefánsson '11, Janson '12, Abraham & Delmas '14] |

| | Global behavior of $\mathcal{T}_n$ | | |
|---|---|---|---|
| | $\mathrm{Var}(\mu) < \infty$ | $\mu(n) \sim \frac{c}{n^{1+\alpha}}$, $\alpha \in (1,2]$ | $\mu(n) \sim \frac{c}{\ln(n)^2 n^2}$, $\alpha = 1$ |
| Critical $\mu$ | <br>Scaling limit:<br>Brownian CRT<br>$\frac{c_\mu}{\sqrt{n}} \cdot \mathcal{T}_n \to \mathcal{T}_e$<br>[Aldous '93] | <br>Scaling limit:<br>$\alpha$-stable tree<br>$\frac{c_\mu}{n^{1-1/\alpha}} \cdot \mathcal{T}_n \to \mathcal{T}_\alpha$<br>[Duquesne '02] | <br>No scaling limits:<br>condensation<br>[K. & Richier '19] |
| Subcritical $\mu$ with $\rho_\mu = 1$ | $\mu(n) \sim \frac{c}{n^{1+\beta}}$, $\beta \geqslant 1$ or $\mu(n) \sim \frac{c}{\ln(n)^2 n^2}$, $\beta = 1$ | | |
| | <br>No scaling limits, condensation<br>[Jonsson & Stefánsson '11, K. '15, K. & Vetter 25'] | | |