Week 9: Binomial coefficients

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Give the "simplest" possible expressions (using as less possible characters as possible) for $\binom{10}{9}, \binom{n}{2}, \binom{3}{5}, \binom{2n}{n}$.

Exercise 2. Consider a group of 10 students. We want to select four people for a team project.

1. How many different teams can be chosen?
2. Assume that there are two members of the group who refuse to work together for this project. How many four-person teams can be formed?

Exercise 3.

1. What is the coefficient of a^2b^2 in the expansion of $(a+b)^4$?
2. What are the coefficients of x^{56}, x^{59} and x^{62} in the expansion of $(x^7+x^{10}+x^{13})^8$?
3. What is the coefficient of $a^2b^5c^3$ in the expansion of $(a+b+c)^{10}$?

Exercise 4. Let $n \geq 6$ be an integer. In how many ways is it possible to place n different balls into n numbered bins in such a way that there are exactly 3 balls in the first bin and exactly 2 balls in the second bin? Give the simplest possible formula.

2 Homework exercise

There are no homework exercises this time ☹.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 9.

Exercise 5. Fix integers $k, n \geq 1$. The goal of this exercise is to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

1. Give an "algebraic" proof.
2. Give a "combinatorial" proof by counting in two different ways the number of subsets of cardinality k of $\{1,2,\ldots,n\}$ having a distinguished element (a "chieftess").

Exercise 6. 1. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, recall the expansion of $(1+x)^n$ as a sum. Deduce simple formulas for $\sum_{k=0}^{n}\binom{n}{k}$ and $\sum_{k=0}^{n} k\binom{n}{k}$.

2. For $n \geq 1$ simplify $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$.

3. Given a set X of cardinality n, show that the number of subsets of X with odd cardinality is equal to the number of subsets of X with even cardinality.

Exercise 7. Let $n \geq 1$ be fixed. What is the largest of the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$?
(Hint: Put $f(k) = \binom{n}{k}$ and compute $f(k+1)/f(k)$.)

Exercise 8. For an integer $n \geq 1$, consider the identity $(\star)$:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

1. Check that $(\star)$ is true for $n = 2$.

2. Consider a group of $2n$ students with n boys and n girls. By counting in two different manners the number of ways to choose group of n students, find a combinatorial proof of $(\star)$.

3. Deduce that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Exercise 9. In how many ways is it possible to put 5 identical coins into 3 different pockets?

Exercise 10. Fix integers $1 \leq m \leq n$.

1) In how many ways can we choose a sequence $(x_1, x_2, \ldots, x_m)$ of nonnegative integers such that $x_1 + x_2 + \cdots + x_m = n$?

2) In how many ways can we choose a sequence $(x_1, x_2, \ldots, x_m)$ of (strictly) positive integers such that $x_1 + x_2 + \cdots + x_m = n$?

Exercise 11. Given integers $0 \leq m \leq n$, show that $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$.

Exercise 12. The game of poker is played with a deck of 52 cards: Deck $= \{1, \ldots, 10, J, Q, K\} \times \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$. A *hand* is a subset of five cards. We consider the following particular hands:

Royal Flush. $10, J, Q, K, 1$ of the same symbol.

Full house. 3 cards of one value, 2 cards of another value. Example : $\{K\heartsuit, 9\clubsuit, 9\heartsuit, K\diamondsuit, 9\spadesuit\}$.

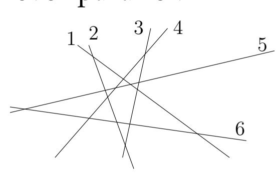
Four of a kind. 4 cards of one value, the fifth card can be any other card. Example : $\{K\heartsuit, 9\clubsuit, 9\heartsuit, 9\diamondsuit, 9\spadesuit\}$.

1. How many different hands of five cards are there?
2. How many hands correspond to a Royal flush?
3. How many hands correspond to a Four of a kind?
4. How many hands correspond to a Full house?

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 9.

Exercise 13. These 6 infinite straight lines are assumed to be in *general position*: three lines never intersect at the same point, and two lines are never parallel.



How many triangles are delimited by these lines ? And what if there are n infinite straight lines in general position ?

For example, the lines 1, 4 and 6 delimate one triangle, which is in bold in the following figure: