Week 6: Notation $\sum$, $\prod$, induction

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $(a_i)_{0\leq i\leq n}$ be a sequence of non-zero real numbers with $a_0 = 0$.

a) What is the value of $\sum_{i=1}^{n}(a_i - a_{i-1})$?

b) What is the value of $\prod_{i=1}^{n-1} \frac{a_{i+1}}{a_i}$?

c) What is the value of $\prod_{i=2}^{n}(1-\frac{1}{i})$?

Solution of exercise 1.

a) This is a telescopic sum, whose value is $a_n - a_0 = a_n$.

b) This is a telescopic product, whose value is $\frac{a_n}{a_1}$.

c) We have $1-\frac{1}{n} = \frac{n-1}{n}$. The product is telescopic, and the product is $\frac{1}{n}$. □

Exercise 2. Show that for every integer $n \geq 1$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution of exercise 2. For $n \geq 1$, let $P(n)$ be the property "$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$".

Basis step. For $n = 1$, we have $1 = 1\cdot 2\cdot 3/6$.

Inductive step. Fix $n \geq 1$ such that $P(n)$ is true. Then

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)(6+7n+2n^2)}{6}.$$

This last quantity is equal to $(n+1)(n+2)(2n+3)/6$, so that $P(n+1)$ is true.

Conclusion. For every $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \tfrac{n(n+1)(2n+1)}{6}.$$

□

Exercise 3. Let $(u_n)_{n\geq 1}$ be the sequence defined by $u_1 = 1$ and for every $n \geq 1$, $u_{n+1} = \frac{u_1+u_2^2+\cdots+u_n^n}{n^n}$. Show that for every $n \geq 1$ we have $0 < u_n \leq 1$.

Solution of exercise 3. We show that $0 < u_n \leq 1$ for every $n \geq 1$ by strong induction on n. To this end, for every $n \geq 1$ let $P(n)$ be the assertion "$0 < u_n \leq 1$".

Basis step. For $n = 1$, we have $u_1 = 1$ and indeed $0 < u_1 \leq 1$.

Inductive step. Fix $n \geq 1$ such that $0 < u_1 \leq 1, \dots, 0 < u_n \leq 1$. We show that $0 < u_{n+1} \leq 1$. First,

$$u_{n+1} = \frac{u_1 + u_2^2 + \cdots + u_n^n}{n^n} \geq \frac{u_1}{n^n} > 0.$$

Second,

$$u_{n+1} = \frac{u_1 + u_2^2 + \cdots + u_n^n}{n^n} \leq \frac{1 + 1 + \cdots + 1}{n^n} = \frac{1}{n^{n-1}} \leq 1.$$

This completes the proof. □

Exercise 4. What do you think of the following reasoning?

Let us show that all sheep in Scotland have the same color.

Basis step. In a set of only one sheep, there is only one color.

Induction step. Assume that within any set of n sheep, there is only one color. Now look at any set of $n+1$ sheep. Number them: $1, 2, \dots, n+1$. Consider the sets $\{1, 2, 3, \dots, n\}$ and $\{2, 3, 4, \dots, n+1\}$. Each is a set of only n sheep, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n+1$ sheep.

Solution of exercise 4. The error comes from the induction step. Indeed, the number n is a fixed number $n \geq 1$. Now each induction step needs the assertion "$\{1, 2, \dots, n\}$ and $\{2, 3, \dots, n+1\}$ overlaps" to be true, however for $n = 1$, the sets $\{1, 2, 3, \dots, n\} = \{1\}$ and $\{2, 3, 4, \dots, n+1\} = \{2\}$ do not overlap, so the reasoning is incorrect.

Conclusion: never write an integer as a "dummy variable" without defining to what set it belongs. □

Exercise 5. Compute $\displaystyle\sum_{1 \leq i, j \leq n} \min(i, j)$.

Solution of exercise 5.

Set $a_n = \sum_{1 \leq i, j \leq n} \min(i, j)$. Then

$$\begin{aligned} a_n &= \sum_{1 \leq i, j \leq n-1} \min(i, j) + 2 \sum_{1 \leq i \leq n-1} \min(i, n) + n \\ &= a_{n-1} + n^2. \end{aligned}$$

Therefore

$$a_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Alternative solution. We have

$$\begin{aligned}
\sum_{1\le i,j\le n} \min(i,j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \min(i,j)\\
&= \sum_{i=1}^{n}\Big(\sum_{j=1}^{i} \min(i,j) + \sum_{j=i+1}^{n} \min(i,j)\Big)\\
&= \sum_{i=1}^{n}\Big(\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i\Big)\\
&= \sum_{i=1}^{n}\Big(\frac{i(i+1)}{2} + i(n-i)\Big)\\
&= -\frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{2n+1}{2}\sum_{i=1}^{n} i\\
&= \frac{n(n+1)(2n+1)}{4} - \frac{n(n+1)(2n+1)}{12}\\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

□

2 Homework exercise

You have to individually hand in the written solution of the next exercises to your TA on Monday, November 18th.

Exercise 6. Let $(b_n)_{n\ge 0}$ be the sequence such that $b_1 = 1$, $b_2 = 3$ and such that for every $n \ge 1$ we have $b_{n+2} = 3b_{n+1} - 2b_n$.

(1) Compute b_3, b_4, b_5.

(2) Propose a simple expression for b_n and prove it.

Solution of exercise 6.

(1) We have $b_3 = 7$, $b_4 = 15$ and $b_5 = 31$.

(2) We show that $b_n = 2^n - 1$ for every $n \ge 1$ by induction on n.

For $n \ge 1$, set $P(n)$: "$b_n = 2^n - 1$ and $b_{n+1} = 2^{n+1} - 1$".

Basis step. We indeed have $b_1 = 1$ and $b_2 = 3$.

Inductive step. Let $n \ge 0$ we such that $P(n)$ is true. We show that $P(n+1)$ is true. First, $b_{n+1} = 2^{n+1} - 1$ since $P(n)$ is true. Second, write

$$b_{n+2} = 3b_{n+1} - 2b_n = 3(2^{n+1}-1) - 2(2^n - 1) = 6\cdot 2^n - 3 - 2\cdot 2^n + 2 = 4\cdot 2^n - 1 = 2^{n+2} - 1.$$

Hence $P(n+1)$ is true. This completes the proof.

Remark. Alternatively one could argue by strong induction. □

Exercise 7. Let $(a_n)_{n\geq 1}$ be the sequence of positive real numbers such that $a_1 = 1$ and such that for every $n \geq 2$,

$$a_n^2 = \sum_{k=1}^{n-1} \frac{a_k}{k}.$$

1) Find the smallest possible value of $c > 0$ such that for every $n \geq 2$, $a_n \leq cn$. Justify your answer.
2) [Optional question] What can you say about a_n as $n \to \infty$?

Solution of exercise 7.

1) The answer is $c = \frac{1}{2}$. Assume first that such a $c > 0$ exists. Since $a_2 = 1$, we get $c \geq \frac{1}{2}$.

Now, let us show that for every $n \geq 2$ we have $a_n \leq n/2$. For $n \geq 2$, let $P(n)$ be the assertion "$a_n \leq n/2$". We show that $P(n)$ is true for every $n \geq 2$ by strong induction on n.

Basis step. We have $a_2 = 1 \leq 2/2$.

Inductive step. Let $n \geq 1$ such that $a_i \leq i/2$ for every $1 \leq i \leq n$. Then

$$a_{n+1}^2 \leq \sum_{k=1}^{n} \frac{k}{2k} = \frac{n}{2}.$$

Hence $a_{n+1} \leq \sqrt{\frac{n}{2}} \leq \frac{n+1}{2}$. Indeed :

$$\sqrt{\frac{n}{2}} \leq \frac{n+1}{2} \iff 2n \leq 1 + 2n + n^2,$$

which is true. Hence $P(n+1)$ is true, which completes the proof.

2) We have $a_n \to \infty$. Indeed, since $a_1 = 1$, we have $a_n^2 \geq 1$ for every $n \geq 1$, so that $a_n \geq 1$ for every $n \geq 1$. This implies that for every $n \geq 2$,

$$a_n^2 \geq \sum_{k=1}^{n-1} \frac{1}{k}.$$

But if $S_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \geq 1$, we have $S_n \to \infty$. Indeed,

$$S_{2n} - S_n = \sum_{k=n+1}^{2n} \frac{1}{k} \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}.$$

Therefore $S_{2^n} \geq \frac{n}{2} \to \infty$, which implies that $S_n \to \infty$ since (S_n) is an increasing sequence.

We conclude that $a_n \to \infty$. □

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 6.

Exercise 8. In Mathland, which is a land where there is $n \in \mathbb{N}$ cities, two cities are always connected either by plane, either by boat (in both directions). Show that it is possible to choose a means of transportation such that starting from any city it is possible to go to any other city by using only the chosen means of transportation.

Solution of exercise 8. Let n be the number of cities. To have an intuition of what is going on, it is always a good idea to see what happens for small values of n. For $n = 2$, there is only one route and one means of transportation. For $n = 3$, let A, B, C be the three cities. By symmetry, let us assume that A and B are connected by plane. Then either C is connected to A or B by plane, in which case one chooses the plane as means of transportation, either C is connected to both A and B by boat, in which case one chooses the boat as means of transportation.

This suggests to show the result by induction. For an integer $n \geq 2$, let P_n be the following assertion:

P_n : "For any configuration of n cities, there exists a means of transportation satisfying the desired conditions."

- Basis step. We have already seen that P_2 is tree.
- Inductive step. Let $n \geq 2$ be an integer and assume that P_n is true. Let us show that P_{n+1} is true. To this end, let us consider a configuration with $n+1$ cities and assume. Let A be any city, and consider the network obtained by removing A and all the transportation lines involving A. We get a configuration with n cities, to which we can apply the property P_n. By symmetry, assume that the plane allows to go from any city to any other in this configuration. There are two cases : either A is connected to some other city by plane, in which case we can choose the plane as means of transportation, either A is connected to all the other cities by boat, in which case we can choose the boat.

□

Exercise 9. A chocolate bar consists of unit squares arranged in an $m \times n$ rectangular grid. You may split the bar into individual unit squares, by breaking along the lines. What is the number of breaks required?

Solution of exercise 9. By testing small cases, it is natural to conjecture that the answer is always $mn - 1$.

When we break the chocolate bar, one of the dimensions decreases, but we do not know in advance which one. For this reason, one way to circumvent this issue is to do an induction on $m + n$. More precisely, for an integer $k \geq 2$, let P_k be the property

P_k : "A chocolate bar with dimensions $m \times n$ satisfying $m + n \leq k$ needs $mn - 1$ breaks".

Basis step. If $m+n \leq 2$, we have $m = n = 1$, and indeed the number of breaks required is $0 = 1 \times 1 - 1$.

Induction step. Assume that $k \geq 2$ is an integer such that P_k is true. Let us show that P_{k+1} is true. To this end, consider a chocolate bar with dimensions $m \times n$ satisfying $m+n \leq k+1$, and do one break.

By symmetry, assume that the chocolate bar is cut into two chocolat bars with dimensions $m_1 \times n$ and $m_2 \times n$ with $1 \le m_1, m_2 \le m-1$ and $m_1 + m_2 = m$. Then

$$m_1 + n = k + 1 - m_2 \le k,$$

so that by the induction hypothesis we need $m_1 n - 1$ breaks to split the first bar into individual squares. Also, $m_2 + n \le k$, so that by the induction hypothesis we need $m_2 n - 1$ breaks to split the second bar into individual squares. Therefore, we need in total

$$1 + (m_1 n - 1) + (m_2 n - 2) = (m_1 + m_2)n - 1 = mn - 1$$

breaks to split the $m \times n$ chocolat bar.

We conclude that the answer is always $mn - 1$. □

Exercise 10. Draw $n \ge 1$ circles in the plane so that two circles are never tangent. Show that using two colours only (for example crimson and teal) it is possible to colour the regions of the planes formed by the circles so that two regions separated by an arc always have different colours.

Solution of exercise 10. We argue by induction. For $n \ge 1$, let $P(n)$ be the property

P_n : "Using crimson and teal, it is possible to colour the regions of the plane formed by n circles so that two regions separated by an arc always have different colours".

Basis step. For $n = 1$, colour the interior of the circle in crimson and the exterior in teal.

Induction step. Assume that $n \ge 1$ is an integer such that $P(n)$ is true. In order to show that $P(n+1)$ is true, consider a configuration with $n+1$ circles and put one aside. In the resulting configuration with n circles, since $P(n)$ is true, we can find of colouring in crimson and in teal of the regions of the plane formed by the n circles so that two regions separated by an arc always have different colours. Now add the circle which has been put aside and perform the following operation : change the colours of all the regions inside that circle. One sees that this colouring meets the required conditions. □

Exercise 11. 73 students travel in a bus with two ticket inspectors. At the beginning, nobody has a ticket, and a passenger only buys a ticket after the third time she is asked to buy a ticket inspector. The ticket inspectors can choose any passenger without a ticket and ask her to buy a ticket. This procedure continues until everyone has a ticket. How many tickets is the first ticket inspector (who is always the one asking first to buy a ticket) sure to sell ?

Solution of exercise 11. The first ticket inspector can sell all the tickets if, at each time its her terme, she proceeds as follows:

- if there is a passenger who has already been chosen twice, she chooses this passenger;
- otherwise, she chooses a passenger who has not yet been chosen.

The first ticket inspector can always proceed according to this rules. Indeed, an induction shows that after the action of the second ticket inspector, the number of persons chosen an even number of times is odd, therefore positive. □

Exercise 12. Recall that $\mathbb{Z}_+ = \{0, 1, 2 \ldots\}$. Let $A \subset \mathbb{Z}_+$ be such that the following two properties hold:

$$1)\ 0 \in A \qquad\qquad 2)\ \forall n \in \mathbb{Z}_+, n \in A \implies n+1 \in A.$$

Show that $A = \mathbb{Z}_+$.

Hint. You may use the (*least integer principle*) following property of $\mathbb{Z}_+$: if $B \subset \mathbb{Z}_+$ is a nonempty subset of $\mathbb{Z}_+$, then B has a smallest element.

Solution of exercise 12. We argue by contradiction and assume that $A \neq \mathbb{Z}_+$ Then the set $B = \mathbb{Z}_+ \setminus A$ is non-empty, and has a smallest element denoted by a. Since $0 \in A$ (by 1)), this means that $0 \notin B$, so that $a \geq 1$. By definition of a, $a-1 \notin B$, so that $a-1 \in A$. But $a-1 \geq 0$. Therefore (by 2) applied with $n = a-1$), $a-1+1 \in A$. Therefore $a \in A$, so that $a \notin B$. This is a contradiction.

Note : this scheme of proof by contradiction using the least integer principle is often called *infinite descent principle (of Fermat)*. □

4 Fun exercise (optional)

The solution of these exercises will be available on the course webpage at the end of week 6.

Exercise 13. Suppose you are informed by your teacher that you will have a test next week, and it will take you by surprise. Then the test can never occur.

Indeed, let us induct backwards.

▹ If it doesn't happen by Thursday, then it must happen on Friday. But that will not be a surprise. So it must happen by Thursday.

▹ If it doesn't happen by Wednesday, then it must happen on Thursday. But that will not be a surprise. So it must happen by Wednesday.

▹ If it doesn't happen by Tuesday, then it must happen on Wednesday. But that will not be a surprise. So it must happen by Tuesday.

▹ If it doesn't happen by Monday, then it must happen on Tuesday. But that will not be a surprise. So it must happen by Monday.

▹ If it happens on Monday, you already predicted it and are not surprised. Hence, the test can never occur.

Do you agree?

Solution of exercise 13. Search for "Unexpected hanging paradox" online. □