Week 3: Quantified Statements

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups A&B, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups C&E, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups D&F, benoit.tran@polytechnique.edu).

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Express the following statements with quantifiers and by using the set A of all the Bachelor students and the assertion $P(x,y)$: "x knows y".

a) Everybody knows everybody, b) Somebody knows everybody, c) There is somebody whom no one knows.

Exercise 2. Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $(u_n)_{n\geq 1}=(u_1,u_2,u_3,\dots)$ be a sequence of real numbers. Write down with quantifiers, and then negate, these propositions:

a) f is bounded above by one, b) f is increasing, c) f is not decreasing.
d) f is increasing and non-negative, e) The terms of the sequence (u_n) are all distinct.
f) The sequence (u_n) is eventually constant.

Remark. In English, by *increasing* and *decreasing* we always mean *strictly*.

Exercise 3. Let A,B be two sets. Write differently, using quantifiers and only the sets A,B, the assertions

a) $A\subset B$ b) $A=B$, c) $A\cap B=\emptyset$.

Write the negation of :

d) $\forall x\in A, x\in B$ e) $\exists x\in A, x\notin B$.

Exercise 4. Prove that $\forall x\in\mathbb{R}, \quad (x=0 \iff \forall\varepsilon>0,\ |x|\leq\varepsilon)$.

Exercise 5. Prove that $\forall\varepsilon>0,\ \exists N>0,\ \forall n\in N,\ \left(n\geq N \implies 1-\varepsilon<\dfrac{n^2+1}{n^2+2}<1+\varepsilon\right)$.

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on October, 14th.

Exercise 6. For $x\in\mathbb{R}$, we define $f(x)=x^2$. Are the follow statements true? Justify your answers

a) $\forall x\in\mathbb{R},\ \exists y\in\mathbb{R},\ y=f(x)$ b) $\exists y\in\mathbb{R},\ \forall x\in\mathbb{R},\ y=f(x)$
c) $\forall y\in\mathbb{R},\ \exists x\in\mathbb{R},\ y=f(x)$ d) $\exists x\in\mathbb{R},\ \forall y\in\mathbb{R},\ y=f(x)$
e) $\forall x\in\mathbb{R},\ \exists M>0,\ f(x)\leq M$ f) $\exists M>0,\ \forall x\in\mathbb{R},\ f(x)\leq M$

Write the negations of the previous statements.

Exercise 7. Show that $\forall x\in\mathbb{R}, (x>0 \iff \exists\varepsilon>0, x\geq\varepsilon)$.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 3.

Exercise 8. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers and $\ell\in\mathbb{R}$. By definition, (u_n) converges to ℓ as $n\to\infty$ if $\forall\varepsilon>0, \exists N>0, \forall n\in\mathbb{N}, (n\geq N \implies |u_n-\ell|<\varepsilon)$.

a) Fix $K \geq 1$. Show that (u_n) converges to ℓ if and only if $\forall \varepsilon > 0, \exists N > K, \forall n \in \mathbb{N}, (n \geq N \implies |u_n - \ell| < \varepsilon)$

b) Assume that (u_n) does not converge to ℓ. Show that there exists $\eta > 0$ and a sequence $(i_n)_{n\geq 1}$ of distinct integers such that $|u_{i_n} - \ell| \geq 2\eta$ for every $n \geq 1$.

Exercise 9. Let A be a set, $x \in A$ and $P(x)$ a proposition which depends on x. Write the assertion $\exists! x \in A, P(x)$ using quantifiers, as well as its negation.

Exercise 10. Colour the points of the plane so that every point is either red, or blue. Show that no matter how the points are coloured the following two properties are true:

(a) for every $x > 0$, there exists a colour C such that there exist two points having colour C and at (Euclidean) distance x.

(b) there exists a colour C such that for every $x > 0$, there exist two points having colour C and at (Euclidean) distance x.

4 Fun exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 3.

Exercise 11. Formalise the following reasonings concerning animals (introduce for example the set K of all kittens and the set N of nice animals, etc.) and say if they are correct.

1) All kittens are nice. But Gizmo is nice. Thus Gizmo is a kitten.

2) Fluffy is a kitten. But all kittens are nice. Thus Fluffy is nice.

3) No kitten is nice. But Spike is not nice. Hence Spike is a kitten.

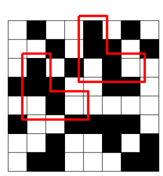
4) No kitten is nice. But Tigger is a kitten. Hence Tigger is not nice.

5) Most of kittens are called Oscar. But all Oscar's are nice. Hence some kittens are nice.

6) All kittens are called Oscar. But some Oscar's are not nice. Hence some kittens are nice.

Exercise 12. Write, using quantifiers: "You may fool all of the people some of the time; you can even fool some of the people all of the time; but you can't fool all of the people all of the time".

Exercise 13. The squares of an 8×8 chessboard are colored black or white. A region composed of 1) a square s 2) the two squares above s 3) the two squares to the right of s is called an "ell". Two "ells" are shown in the figure.



Prove that no matter how we color the chessboard, there must be two "ells" that are colored identically (as illustrated in the figure).