Week 3: Quantified Statements

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Express the following statements with quantifiers and by using the set A of all the Bachelor students and the assertion $P(x,y)$: "x knows y".

a) Everybody knows everybody, b) Somebody knows everybody, c) There is somebody whom no one knows.

Solution of exercise 1.

a) $\forall x \in A, \forall y \in A, P(x,y)$.

b) $\exists x \in A, \forall y \in A, P(x,y)$.

c) $\exists x \in A, \forall y \in A, \neg P(y,x)$ □

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $(u_n)_{n\geq 1} = (u_1, u_2, u_3, \dots)$ be a sequence of real numbers. Write down with quantifiers, and then negate, these propositions:

a) f is bounded above by one, b) f is increasing, c) f is not decreasing.
d) f is increasing and non-negative, e) The terms of the sequence (u_n) are all distinct.
f) The sequence (u_n) is eventually constant.

Remark. In English, by *increasing* and *decreasing* we always mean *strictly*.

Solution of exercise 2.

a) $\forall x \in \mathbb{R}, f(x) \leq 1$. The negation is $\exists x \in \mathbb{R}, f(x) > 1$.

b) $\forall x, y \in \mathbb{R}, x < y \implies f(x) < f(y)$.

The negation is $\exists x, y \in \mathbb{R}, x < y$ and $f(x) \geq f(y)$.

c) Let's start with writing the negation, that is f is decreasing. This is $\forall x, y \in \mathbb{R}, x < y \implies f(x) > f(y)$.

f is not strictly decreasing is therefore: $\exists x, y \in \mathbb{R}, x < y$ and $f(x) \leq f(y)$.

d) f is increasing and non-negative : $\big(\forall x, y \in \mathbb{R}, x < y \implies f(x) < f(y)\big) \wedge (\forall x \in \mathbb{R}, f(x) \geq 0)$.

Negation:

$$\big(\exists x, y \in \mathbb{R}, x < y \text{ and } f(x) \geq f(y)\big) \text{ or } (\exists x \in \mathbb{R}, f(x) < 0).$$

e) $\forall i, j \in \mathbb{N}, i \neq j \implies u_i \neq u_j$. The negation is

$$\exists i, j \in \mathbb{N}, i \neq j \text{ and } u_i = u_j.$$

f) $\exists N \geq 1, \forall n \in \mathbb{N}, n \geq N \implies u_n = u_N$.

Negation:

$$\forall N \in \mathbb{N}, \exists n \geq N, u_n \neq u_N.$$

$\square$

Exercise 3. Let A, B be two sets. Write differently, using quantifiers and only the sets A, B, the assertions

a) $A \subset B$ b) $A = B$, c) $A \cap B = \emptyset$.

Write the negation of :

d) $\forall x \in A, x \in B$ e) $\exists x \in A, x \notin B$.

Solution of exercise 3.

a) $\forall x \in A, x \in B$.

b) $(\forall x \in A, x \in B) \wedge (\forall x \in B, x \in A)$.

c) $\forall x \in A, x \notin B)$ *Remark.* $\forall x \in B, x \notin A$ works as well.

d) $\exists x \in A, x \notin B$

e) $\forall x \in A, x \in B$.

$\square$

Exercise 4. Prove that $\forall x \in \mathbb{R}, \quad (x = 0 \iff \forall \varepsilon > 0, \ |x| \leq \varepsilon)$.

Solution of exercise 4. The "$\implies$" part is clear. Indeed, fix $x \in \mathbb{R}$ and assume that $x = 0$. Fix $\varepsilon > 0$. The statement $|0| \leq \epsilon$ is true. This shows that when $x = 0$, $\forall \varepsilon > 0, \ |x| \leq \varepsilon$ is true.

Let us prove "$\impliedby$".

First method. Fix $x \in \mathbb{R}$. We show that the contrapositive $x \neq 0 \implies \exists \epsilon > 0, |x| > \varepsilon$. To show this assertion, assume that $x \neq 0$. It is enough to find $\varepsilon > 0$ such that $|x| > \epsilon$. We may take for example $\varepsilon = |x|/2$.

Second method. Fix $x \in \mathbb{R}$ such that $\forall \varepsilon > 0, |x| \leq \varepsilon$. We argue by contradiction and we assume that $x \neq 0$. But we have

$$\frac{|x|}{2} < |x|.$$

Then we have found a **particular** $\varepsilon > 0$ such that $\varepsilon < |x|$. This contradicts the fact that **for all** $\varepsilon > 0$ we have $|x| \leq \varepsilon$. $\square$

Exercise 5. Prove that $\forall \varepsilon > 0, \ \exists N > 0, \ \forall n \in N, \ \left(n \geq N \implies 1 - \varepsilon < \frac{n^2+1}{n^2+2} < 1 + \varepsilon \right)$.

Solution of exercise 5. We fix $\varepsilon > 0$. We need to prove the existence of $N > 0$ such that for $n \geq N$we have

$$1 - \varepsilon < \frac{n^2+1}{n^2+2} < 1 + \varepsilon.$$

In other words, we have to show that these inequalities hold for every n large enough.

Upper bound. For $n \geq 1$, since $n^2+1 \leq n^2+2$, we always have

$$\frac{n^2+1}{n^2+2} \leq 1 < 1+\varepsilon.$$

Lower bound. We write, for $n \geq 1$.

$$\frac{n^2+1}{n^2+2} = \frac{n^2+2}{n^2+2} - \frac{1}{n^2+2} = 1 - \frac{1}{n^2+2} \geq 1 - \frac{1}{n^2}.$$

Now, if $1/n^2 \leq \varepsilon$, *i.e.* $n > \sqrt{1/\varepsilon}$ then the right-hand side is greater that $1-\varepsilon$.
Conclusion. For every $\varepsilon > 0$, there exists $N_\varepsilon = \lfloor \sqrt{1/\varepsilon} \rfloor + 1$ (where $\lfloor x \rfloor$ is the integer part of x, *i.e.* the greatest integer smaller than or equal to x) such that

$$n \geq N_\varepsilon \implies 1-\varepsilon < \frac{n^2+1}{n^2+2} < 1+\varepsilon.$$

□

2 Homework exercises

You have to individually hand in the written solution of the next exercises to your TA on October, 14th.

Exercise 6. For $x \in \mathbb{R}$, we define $f(x) = x^2$. Are the follow statements true? Justify your answers

a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y = f(x)$ b) $\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ y = f(x)$
c) $\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ y = f(x)$ d) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y = f(x)$
e) $\forall x \in \mathbb{R},\ \exists M > 0,\ f(x) \leq M$ f) $\exists M > 0,\ \forall x \in \mathbb{R},\ f(x) \leq M$

Write the negations of the previous statements.

Solution of exercise 6.

a) True: for fixed $x \in \mathbb{R}$, we take $y = x^2$.

b) False: argue by contradiction and assume that $y \in \mathbb{R}$ is such that $y = x^2$ for every $x \in \mathbb{R}$. By taking $x = 1$, we obtain $y = 1$. By taking $x = 2$, we obtain $y = 4$. This is a contradiction.

c) False: if $y = -1$, we cannot find $x \in \mathbb{R}$ such that $-1 = x^2$.

d) False: argue by contradiction and assume that there is $x \in \mathbb{R}$ such that $y = x^2$ for every $y \in \mathbb{R}$. By taking $y = -1$ we obtain a contradiction.

e) True: for fixed $x \in \mathbb{R}$, we take $M = x^2$.

f) False. We argue by contradiction and assume that there exists $M > 0$ such that for every $x \in \mathbb{R}$ we have $x^2 \leq M$. We take $x = M+1$ and obtain $(M+1)^2 \leq M$, that is $M^2+M+1 \leq 0$, which is a contradiction.

The negations are

a) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y \neq f(x)$ b) $\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ y \neq qf(x)$
c) $\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ y \neq f(x)$ d) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y \neq f(x)$
e) $\exists x \in \mathbb{R},\ \forall M > 0,\ f(x) > M$ f) $\forall M > 0,\ \exists x \in \mathbb{R},\ f(x) > M$

□

Exercise 7. Show that $\forall x \in \mathbb{R}, (x > 0 \iff \exists \varepsilon > 0, x \geq \varepsilon)$.

Solution of exercise 7. Fix $x \in \mathbb{R}$. We argue by double inclusion.

We first show $\Longrightarrow$. Assume that $x > 0$. Then by taking $\varepsilon = x/2$, we indeed have $\varepsilon > 0$ and $x \geq \varepsilon$.

We next show $\Longleftarrow$. Assume that $\exists \varepsilon > 0$ such that $x \geq \varepsilon$. Then $x \geq \varepsilon 0$.

□

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 3.

Exercise 8. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers and $\ell \in \mathbb{R}$. By definition, (u_n) converges to ℓ as $n \to \infty$ if $\forall \varepsilon > 0, \exists N > 0, \forall n \in \mathbb{N}, (n \geq N \implies |u_n - \ell| < \varepsilon)$.

a) Fix $K \geq 1$. Show that (u_n) converges to ℓ if and only if $\forall \varepsilon > 0, \exists N > K, \forall n \in \mathbb{N}, (n \geq N \implies |u_n - \ell| < \varepsilon)$

b) Assume that (u_n) does not converge to ℓ. Show that there exists $\eta > 0$ and a sequence $(i_n)_{n\geq 1}$ of distinct integers such that $|u_{i_n} - \ell| \geq 2\eta$ for every $n \geq 1$.

Solution of exercise 8. a) Let us first show the direct implication. Assume that (u_n) converges to ℓ. Fix $\epsilon > 0$ and $N > 0$ such that if $n \geq N$, then $|u_n - \ell| < \varepsilon$. It follows that if $n \geq \max(N,K)+1$, then $|u_n - \ell| < \varepsilon$). There, we have found an integer greater than K, namely $\max(N,K)+1$, such that $\forall n \in \mathbb{N}, (n \geq \max(N,K)+1 \implies |u_n - \ell| < \varepsilon)$.

Let us now show the converse implication. Fix $\epsilon > 0$ and $N > K$ such that if $n \geq N$, then $|u_n - \ell| < \varepsilon$. In particular, this means that there exists exists an integer $N > 0$ such that if $n \geq N$, then $|u_n - \ell| < \varepsilon$, which shows that (u_n) converges to ℓ as $n \to \infty$.

b) Since (u_n) does not converge to ℓ, this means that $\exists \varepsilon > 0$, $\forall N > 0$, $\exists n \geq N$ and $|u_n - \ell| \geq \varepsilon$. Therefore, we can fix $\epsilon > 0$ such that the following assertion is true:

(P) $\qquad \forall N > 0, \exists n \geq N$ and $|u_n - \ell| \geq \varepsilon$.

We use (P) with $N = 1$: we can find $i_1 \geq 1$ such that $|u_{i_1} - \ell| \geq \varepsilon$.

We use (P) with $N = i_1 + 1$: we can find $i_2 > i_1$ such that $|u_{i_2} - \ell| \geq \varepsilon$.

We use (P) with $N = i_2 + 1$: we can find $i_3 > i_2$ such that $|u_{i_3} - \ell| \geq \varepsilon$.

We continue in this manner and we construct a sequence $(i_n)_{n\geq 1}$ of distinct integers such that $|u_{i_n} - \ell| \geq \varepsilon$ for every $n \geq 1$.

This completes the proof by taking $\eta = \epsilon/2$.

□

Exercise 9. Let A be a set, $x \in A$ and $P(x)$ a proposition which depends on x. Write the assertion $\exists! x \in A, P(x)$ using quantifiers, as well as its negation.

Solution of exercise 9. Saying that $\exists! x \in A, P(x)$ is true is saying that there exists $x \in A$ such that $P(x)$ is true, and that if $P(x)$ and $P(y)$ are true, then $x = y$.

$$(\exists x \in A, P(x)) \wedge \big(\forall x, y \in A, (P(x) \wedge P(y) \implies x = y)\big).$$

Its negation is

$$(\forall x \in A, \neg P(x)) \vee \big(\exists x, y \in A, P(x) \wedge P(y) \wedge (x \neq y)\big).$$

□

Exercise 10. Colour the points of the plane so that every point is either red, or blue. Show that no matter how the points are coloured the following two properties are true:

(a) for every $x > 0$, there exists a colour C such that there exist two points having colour C and at (Euclidean) distance x.

(b) there exists a colour C such that for every $x > 0$, there exist two points having colour C and at (Euclidean) distance x.

Solution of exercise 10.

(a) Fix $x > 0$. Consider an equilateral triangle ABC with side lengths x. By symmetry, assume that A is red. If B or C are red, we choose $C =$ "red". If B and C are blue, we choose $C =$ "blue".

(b) The main difference with the previous question is that now the choice of the colour C has to be the same for all choices of x. To this end, we argue by contradiction. This means that there exists $x_{\text{blue}} > 0$ such that two points at distance x_{blue} are not both blue, and that there exists $x_{\text{red}} > 0$ such that two points at distance x_{red} are not both red.

There exists a red point A. Indeed, otherwise, all the points would be blue, and this would contradict the fact that two points at distance x_{blue} are not both blue. Now consider an isosceles triangle ABC such that $AB = AC = x_{\text{red}}$ and $BC = x_{\text{blue}}$. Then B and C must be blue. But they are at distance x_{blue}. This is a contradiction, and completes the proof.

□

4 Fun exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 3.

Exercise 11. Formalise the following reasonings concerning animals (introduce for example the set K of all kittens and the set N of nice animals, etc.) and say if they are correct.

1) All kittens are nice. But Gizmo is nice. Thus Gizmo is a kitten.
2) Fluffy is a kitten. But all kittens are nice. Thus Fluffy is nice.
3) No kitten is nice. But Spike is not nice. Hence Spike is a kitten.
4) No kitten is nice. But Tigger is a kitten. Hence Tigger is not nice.
5) Most of kittens are called Oscar. But all Oscar's are nice. Hence some kittens are nice.
6) All kittens are called Oscar. But some Oscar's are not nice. Hence some kittens are nice.

Solution of exercise 11.

1) Not correct: we assume that $K \subset N$ and Gizmo $\in N$. We cannot conclude that Gizmo $\in K$ since N could have more elements than K.

2) Correct: we assume Fluffy $\in K$ and $K \subset N$. Hence Fluffy $\in N$ so we can conclude that Fluffy is nice.

3) Not correct: we assume $K \cap N = \emptyset$ and that Spike $\notin N$. We cannot conclude that Spike $\in K$ (Spike could be another animal, we could have concluded that Spike $\in K$ if we knew that $K \cup N$ is the set of all animals).

4) Correct: we assume $K \cap N = \emptyset$ and Tigger $\in K$. Hence Tigger $\notin N$, so we can conclude that Tigger is not nice.

5) Correct. Let O be the set of all animals called Oscar. "Most of" has no precise mathematical sense. A plausible way is to assume that $K \cap O \neq \emptyset$, $O \subset N$. A plausible way to interpret "some kittens are nice" is $K \cap N \neq \emptyset$. This is indeed true. Indeed, if we take $x \in K \cap O$, we hence $x \in K$ and $x \in O$, hence $x \in K$ and $x \in N$, hence $K \cap N \neq \emptyset$.

6) Not correct. Let O be the set of all animals called Oscar. We assume that $K \subset O$ and that $O \cap \overline{N} \neq \emptyset$. We cannot conclude that $K \cap N \neq \emptyset$. Indeed, if all Oscars are not nice (that is $O \subset \overline{N}$) we have $K \cap N = \emptyset$, but if there are two oscar kittens, one nice and one not nice, we have $K \cap N \neq \emptyset$. □

Exercise 12. Write, using quantifiers: "You may fool all of the people some of the time; you can even fool some of the people all of the time; but you can't fool all of the people all of the time".

Solution of exercise 12. Let A be the set of all the people and T be the set of all times.

If one interprets "some" by "there exists", this assertion is

$$\Big(\exists t \in T, \forall x \in A, x \text{ is fooled at time } t\Big) \wedge \Big(\exists x \in A, \forall t \in T\, x \text{ is fooled at time } t\Big)$$
$$\wedge \neg\Big(\forall t \in T, \forall x \in A, x \text{ is fooled at time } t\Big),$$

which is logically equivalent to

$$\Big(\exists t \in T, \forall x \in A, x \text{ is fooled at time } t\Big) \wedge \Big(\exists x \in A, \forall t \in T\, x \text{ is fooled at time } t\Big)$$
$$\wedge\Big(\exists t \in T, \exists x \in A, x \text{ is not fooled at time } t\Big),$$

If one interprets "some" by "at least two", this assertion is

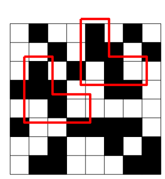
$$\Big(\exists s, t \in T, s \neq t \text{ and } \forall x \in A, x \text{ is fooled at time } t\Big)$$
$$\wedge\Big(\exists x, y \in A, x \neq y \text{ and } \forall t \in T, x \text{ and } y \text{ are fooled at time } t\Big) \wedge \neg\Big(\forall t \in T, \forall x \in A, x \text{ is fooled at time } t\Big),$$

which is logically equivalent to

$$\Big(\exists s, t \in T, s \neq t \text{ and } \forall x \in A, x \text{ is fooled at time } t\Big)$$
$$\wedge\Big(\exists x, y \in A, x \neq y \text{ and } \forall t \in T, x \text{ and } y \text{ are fooled at time } t\Big) \wedge \Big(\exists t \in T, \exists x \in A, x \text{ is not fooled at time } t\Big),$$

□

Exercise 13. The squares of an 8×8 chessboard are colored black or white. A region composed of 1) a square s 2) the two squares above s 3) the two squares to the right of s is called an "ell". Two "ells" are shown in the figure.



Prove that no matter how we color the chessboard, there must be two "ells" that are colored identically (as illustrated in the figure).

Solution of exercise 13. First, there are $2^5 = 32$ ways to color a given "ell" in black/white. Second, there are $6 \times 6 = 36$ distinct ells in a 8×8 chessboard (6 possible abscissa and 6 possible ordinates for its "main square"). Therefore (by the pigeonhole principle) there must be two ells colored identically. $\square$