# Week 2, September 30: Formal logic, truth tables, methods of proof

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups $A$&$B$, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups $C$&$E$, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups $D$&$F$, benoit.tran@polytechnique.edu).

## 1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

*Exercise 1.* If $x, y \in \mathbb{R}$ are real numbers, write the negation of the following assertions:

a) $x = 0$  b) $x \geq 0$  c) $|x| < 1$  d) ($x \geq 0$ and $y \geq 0$) or ($x \leq 0$ and $y \leq 0$).

*Exercise 2.* When $Y, Z$ are sets, write the contrapositive of $(x \in Y) \implies (x \in Z) \vee (x \notin Y)$.

*Exercise 3.* In each case, are the two assertions logically equivalent?

a) $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$  b) $P \vee Q$ and $(P \wedge (\neg Q)) \vee ((\neg P) \wedge Q)$.

*Exercise 4.* Which of the following statements are true? Justify your answer.

a) $(\neg(\emptyset = \emptyset)) \wedge (\neg(\emptyset \in \emptyset))$  b) $(\emptyset \in \emptyset) \implies (\emptyset \subseteq \emptyset)$  c) $(\emptyset = \emptyset) \vee (\neg(\emptyset \in \emptyset))$.

*Exercise 5.* If $A, B$ are subsets of a set $E$, write the negation of the assertion "$x \in \overline{A} \cap B$" in the form "$x \in \ldots$".

*Exercise 6.* When $P$ and $Q$ are assertions, write a logical formula for the following phrases:

a) $P$ but not $Q$  b) either $P$ or $Q$ but not both  c) $P$, unless $Q$.

Also give their negations.

*Exercise 7.* Write the following statement in if-then form: "Every perfect integer is even" (you do not need to know what a "perfect" integer is to solve the exercise).

*Exercise 8.* Let $x$ and $y$ be two integers. Show that $x + y$ is even if and only if $x$ and $y$ have the same parity.

*Exercise 9.* Around 546 BC, Croesus (595 BC – 546 BC) began preparing a campaign against Cyrus the Great of Persia. He turned to the Delphic oracle for advice. The oracle said that if Croesus attacked the Persians, he would destroy a great empire. Croesus attacked the Persians, but was eventually defeated by Cyrus. Was the oracle wrong?

*Exercise 10.* During the first world war, someone noticed that the planes that had returned to base had impacts everywhere, except on the cockpit. She deduced that the cockpit was fragile and had to be reinforced. Do you agree?

*Exercise 11.* What do you think of the following argumentation?

"All impartial observers and all credible theorists hold that when the basic structures of a society are fair, citizens conform to them of their own will. The fact that citizens in our societies do not rebel thus constitutes a powerful and convincing proof of the justice of our basic institutions, and all our so-called revolutionaries would be wise to think about that carefully."

## 2 Homework exercises

You have to individually hand in the written solutions of the next two exercises to your TA on October, 7th.

*Exercise 12.* Let $a, b$ be two positive integers.

1) Show that $a^2 > b^2 + 1 \implies a \geq b + 1$.

2) Is the implication $a^2 > b^2 + 1 \implies a \geq b + 1$ always true if $a, b$ are integers (not necessarily positive)?

*Exercise 13.* Let $A$ and $B$ be two sets. Show that $A \subseteq B \iff A \cap B = A$.

## 3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 2.

*Exercise 14.* Below are 5 statements (in "formal logic" language). How many of them can be true together?

(a): if (b) is true then (a) is false.

(b): if number of the true statements is (strictly) greater than 2, then one of them is ($c$).

(c): at least one of (a) and (d) is false.

(d): (b) and (c) are both true or both false.

(e): (b) is true or false.

*Exercise 15.* Define the median of a set of $k$ numbers as follows: first put the numbers in non-decreasing order; then the median is the middle number if $k$ is odd, and the average of the two middle numbers if $k$ is even. (For example, the median of $\{1, 3, 4, 8, 9\}$ is 4, and the median of $\{1, 3, 4, 7, 8, 9\}$ is $(4 + 7)/2 = 5.5$.)

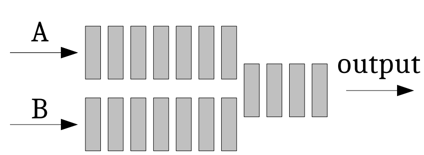
Let $n \geq 1$ be a positive integer, and define $S_n = \{1, 2, ..., n\}$. If $T$ is a non-empty subset of $S_n$, we say that $T$ is balanced if the median of $T$ is equal to the average of $T$. For example, for $n = 9$, each of the subsets $\{7\}, \{2, 5\}, \{2, 3, 4\}, \{5, 6, 8, 9\}$, and $\{1, 4, 5, 7, 8\}$ is balanced; however, the subsets $\{2, 4, 5\}$ and $\{1, 2, 3, 5\}$ are not balanced.

Prove that the number of balanced subsets of $S_n$ is odd.

## 4 Fun exercises (optional)

The solution of this exercise will be available on the course webpage at the end of week 2.

*Exercise 16.* **A computer with dominos.** A logic gate is the most elementary block of a digital circuit. For instance, the "or" gate passes on a signal if at least one of the inputs is on. It is possible to design logic gates with dominos: the gate passes on the signal if the chain of dominos is falling. With dominos the "or" gate is obtained by:




Can you make the gate "$A$ and (not $B$)"? And the "and" gate?