Week 15: Review exercises

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1 Some review exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Complete the boxes $\boxed{}$ with $\Longrightarrow$, $\Longleftarrow$, or $\Longleftrightarrow$ and justify your answers:

a) Let $f : E \to F$ be a function. Then $\forall x, y \in E$, $x = y$ $\boxed{}$ $f(x) = f(y)$

b) Let $f : E \to F$ be a one-to-one function. Then $\forall x, y \in E$, $x = y$ $\boxed{}$ $f(x) = f(y)$

c) Let $f : E \to F$ be an onto function. Then $\forall x, y \in E$, $x = y$ $\boxed{}$ $f(x) = f(y)$

d) Let $f : E \to F$ be a bijective function. Then $\forall x, y \in E$, $x = y$ $\boxed{}$ $f(x) = f(y)$

e) Let $f, g : E \to F$ be functions. Then $f = g$ $\boxed{}$ $f(E) = g(E)$.

Solution of exercise 1.

a) $\boxed{\Longrightarrow}$. The implication is always true by the definition of a function. The converse is not true in general (it is false if the function is not one-to-one).

b) $\boxed{\Longleftrightarrow}$. The implication is always true by the definition of a function. The converse is true since f is one-to-one.

c) $\boxed{\Longrightarrow}$. The implication is always true by the definition of a function. The converse is not true in general (it is false if the function is not one-to-one).

d) $\boxed{\Longleftrightarrow}$. The implication is always true by the definition of a function. The converse is true since f is one-to-one.

e) $\boxed{\Longrightarrow}$. Indeed, if $f = g$, then $f(x) = g(x)$ for every $x \in E$, so $f(E) = g(E)$. The converse is false (take $E = \{1,2,3\}$, $F = \{1,2\}$, f defined by $f(1) = f(2) = 1$ and $f(3) = 2$, g defined by $g(1) = 1$ and $g(2) = g(3) = 2$). $\square$

Exercise 2. For $n \geq 1$, consider the permutation $\begin{array}{cccc} \sigma : & \{1,2,\ldots,n\} & \to & \{1,2,\ldots,n\} \\ & i & \mapsto & n+1-i \end{array}$

1) Write the cycle decomposition of σ.

2) What is the value of $\varepsilon(\sigma)$?

Solution of exercise 2.

1. If $n = 2k$ is even, $\sigma = (1,n) \circ (2, n-1) \cdots \circ (k, k+1)$ (with k factors). If $n = 2k+1$ is odd, $\sigma = (1,n) \circ (2, n-1) \cdots \circ (k, k+2)$ (with k factors).

2. By the previous question, using the multiplicativity property of the signature and the fact that

the signature of a transposition is -1, we have $\varepsilon(\sigma) = (-1)^{n/2}$ if n is even and $\varepsilon(\sigma) = (-1)^{(n-1)/2}$ if k is odd.

(In other words, $\varepsilon(\sigma) = 1$ if $n \equiv 0, 1 \mod 4$ and $\varepsilon(\sigma) = -1$ if $n \equiv 2, 3 \mod 4$.)

Remark. It is possible to give an alternative solution by computing the number of inversions of σ, which is $(n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2}$. □

Exercise 3. Six people each throw a fair dice.

a) What is the probability that there are exactly three sixes?

b) What is the probability that the largest number that appears is at least 4?

c) What is the probability that the largest number that appears is exactly 3?

Solution of exercise 3. We take $\Omega = \{1, 2, 3, 4, 5, 6\}^6$ (which has 6^6 elements), equipped with the uniform probability measure.

a) We count the number of configurations where there are exactly three sixes. We have $\binom{6}{3}$ choices for the people who get a 6. The other three people then get 1, 2, 3, 4 or 5, which gives 5^3 choices. We conclude that

$$\mathbb{P}(\text{there are exactly three sixes}) = \frac{\binom{6}{3} \times 5^3}{6^6}.$$

b) We use the complementary event and compute rather the probability that the largest number is at most 3. To this end, we count the number of configurations where the largest number is at most 3. In this case, each dice gives 1, 2 or 3, so the latter probability is $\frac{3^6}{6^6} = \frac{1}{2^6}$. Therefore the probability that the largest number that appears is at least 4 is $1 - \frac{1}{2^6}$.

c) Let E be the event "the largest number that appears is exactly 3". The idea is to write the complementary event $\overline{E}$ as the disjoint union of two events A and B where :

– A is the event "the largest number that appears is at most 2"

– B is the event "the largest number that appears is at least 4"

so that $1 - \mathbb{P}(E) = \mathbb{P}(A) + \mathbb{P}(B)$.

By question b), we have $\mathbb{P}(B) = 1 - \frac{1}{2^6}$. The same reasoning as in question b) gives that $\mathbb{P}(A) = \frac{1}{3^6}$. Hence

$$1 - \mathbb{P}(E) = 1 - \frac{1}{2^6} + \frac{1}{3^6},$$

so we conclude that

$$\mathbb{P}(E) = \frac{1}{2^6} - \frac{1}{3^6}.$$

□

Exercise 4. What is the coefficient of x^8 in the expansion of $(1 + x^2 + x^3)^{40}$ (you can leave binomial coefficients)? Justify your answer.

Solution of exercise 4. Since $8 = 2 + 2 + 2 + 2$ and $8 = 2 + 3 + 3$, in the expansion of $(1 + x^2 + x^3)^{40}$ we have two possibilities to obtain x^8:

- taking x^2 four times and 1 thirty-six times
- taking x^2 once, x^3 twice, 1 thirty-seven times.

Finally, the answer is $\binom{40}{4}+40\times\binom{39}{2}$ (which is the same as $\binom{40}{4}+\binom{40}{2}\times 38$). □

Exercise 5. Let $f : E \to F$ be a function. Show that f is one-to-one if and only if $\forall A \subseteq E, A = f^{-1}(f(A))$.

Solution of exercise 5. We argue by double implication. First assume that f is one-to-one. We fix $A \subseteq E$ and show that $A = f^{-1}(f(A))$ by double inclusion.

First take $x \in A$. Thus $f(x) \in f(A)$ which is equivalent to $x \in f^{-1}(f(A))$.

Second take $x \in f^{-1}(f(A))$. Hence $f(x) \in f(A)$. This means that there exists $y \in A$ such that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y \in A$.

We show the converse by contraposition. Assume that f is not one-to-one. We shall show that there exists $A \subseteq E$ such that $A \neq f^{-1}(f(A))$. Since f is not one-to-one, we may find $x \neq y$ such that $f(x) = f(y)$. Take $A = \{x\}$. Then $\{x, y\} \subseteq f^{-1}(f(A))$ (because $f(x) \in f(A)$ and $f(y) = f(x) \in f(A)$. Therefore $A \neq f^{-1}(f(A))$.

Alternative proof for the converse implication. Assume that $\forall A \subseteq E, A = f^{-1}(f(A))$. Let $x, y \in E$ be such that $f(x) = f(y)$. Then we have that

$$f^{-1}(\{f(x)\}) = f^{-1}(\{f(y)\}) \Leftrightarrow f^{-1}(f(\{x\})) = f^{-1}(f(\{y\})) \Leftrightarrow \{x\} = \{y\} \Leftrightarrow x = y.$$

□