Week 13: Combinatorics and probability

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1 Exercises

The solutions of the questions which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $n \geq 0$ be an integer. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Exercise 2. Let $n \geq 1$ be an integer.

1) We want to choose a subset of $\{1, 2, \ldots, n\}$ uniformly at random. Give a probability space to model this experiment and compute the probability of the following two events:

a) "the subset has cardinality 1".

b) "the subset contains 1"

2) We throw a dice with six faces which is not a fair dice, such that the following condition (H) is satisfied:

(H) : "the probability of falling on a face is proportional to its value."

Give a probability space $(\Omega, \mathbb{P})$ to model this experiment, write what condition (H) means using $\mathbb{P}$ and compute the probability that the dice falls on 6.

Exercise 3. Let $n \geq 1$ be an integer. The goal of this exercise is to study partitions of the set $\{1, 2, \ldots, n\}$. By definition, a partition of $\{1, 2, \ldots, n\}$ is a set of nonempty subsets of $\{1, 2, \ldots, n\}$ which are pairwise disjoint and whose union is $\{1, 2, \ldots, n\}$. We say that a partition is a k-partition if it has cardinality k.

For example, $\{\{1, 8\}, \{2, 3, 4, 5, 6, 9\}, \{7\}\}$ is a 3-partition of $\{1, 2, \ldots, n\}$.

Denote by $B_{n,k}$ the total number of k-partitions of $\{1, 2, \ldots, n\}$ and denote by B_n the total number of partitions of $\{1, 2, \ldots, n\}$.

1) Write the set of all partitions of $\{1, 2, 3\}$.

2) Give the values of B_1, B_2, B_3.

3) What is the value of $B_{n,n-1}$?

4) What is the value of $B_{n,2}$?

5) Fix an integer $1 \leq k \leq n$.

a) Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ be an onto map. For $1 \leq i \leq k$, set $A_i = f^{-1}(\{i\})$. Show that $\{A_1, A_2, \ldots, A_k\}$ is a k-partition of $\{1, 2, \ldots, n\}$.

b) Let $S_{n,k}$ be the number of onto maps from $\{1, 2, \ldots, n\}$ to $\{1, \ldots, k\}$. Show that $S_{n,k} = k! \times B_{n,k}$.

6) Set $B_0 = 1$. Show that for $n \geq 2$, $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$.

Remark. This formula gives a recursive way to compute the value of B_n. There is no simple expression for B_n.

2 Homework exercise

Exercise 4. Let $n \geq 1$ be an integer. We want to choose a subset of $\{1, 2, \ldots, n\}$ at random in such a way that the following condition (C) is satisfied:

(C) "there exists a value $a > 0$ such that the probability of choosing a subset containing 1 is a and the probability of choosing a subset not containing 1 is $2a$".

1) Give a probability space $(\Omega, \mathbb{P})$ to model this experiment, and write what condition (C) means using $\mathbb{P}$.

2) Give a simple expression of a involving only n, and justify your answer.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 13.

Exercise 5. For $n \geq 1$, we call a *path* of length $2n$ any sequence $y_0, y_1, y_2, \ldots, y_{2n}$ of integers such that $y_0 = 0$ and for every $1 \leq i \leq 2n$, $y_i - y_{i-1} \in \{-1, +1\}$.

1) How many paths of length $2n$ are there ?

2) A path is called a *bridge* if $y_{2n} = 0$. How many bridges of length $2n$ are there?

Exercise 6. Let $1 \leq p \leq n$ be integers. Let E be a set with n elements and A a subset of E with p elements.

1) How many subsets X of E such that $A \subset X$ are there?

2) If $p \leq m \leq n$, how many subsets X of E such that $A \subset X$ are there?

3) How many couples (X, Y) of subsets of E such that $X \cap Y = A$ are there?

Exercise 7. Let $n \geq 2$ be an integer and let us consider a deck of n cards numbered from 1 to n.

1) In how many ways is it possible to shuffle the deck so that the card with number 1 is further in the deck than the card 2?

2) In how many ways is it possible to shuffle the deck so that the cards with numbers 1 and 2 are neighbours ?

Exercise 8. Show that for every $n \geq 1$, $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 13.

Exercise 9. 71 mathematicians are standing in a line, wearing a black or white hat. Each mathematician can ONLY see the color of the hats of the people in front of them. So the first person sees no hats, the last sees 70. The mathematicians are allowed to talk to each other and decide upon a strategy, for a government rep is coming to cut off funding. Each person can only say "black" or "white." If you correctly say the color of the hat you're wearing, your funding is continued and you live. If you're wrong, you lose your funding, and you may as well be dead.

How many mathematicians can you guarantee will keep their funding?

(You are not allowed to use "tricks," say a person delays one second before answering means A, two seconds means B, ... You have to answer IMMEDIATELY what color hat you're wearing.)