Week 13: Combinatorics and probability

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1 Exercises

The solutions of the questions which have not been solved in some group will be available on the course webpage.

Exercise 1. Let $n \geq 0$ be an integer. Show that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Solution of exercise 1. *First solution (algebraic).* Simply apply the binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

Second solution (combinatorial). We count in two ways the number of subsets of a set of size n. We already know that it is equal to 2^n. A subset can also be constructed as follows: first fix its size $0 \leq k \leq n$ and then choose the k elements among n which constitute the set ($\binom{n}{k}$ possibilities). This shows the desired formula.

□

Exercise 2. Let $n \geq 1$ be an integer.

1) We want to choose a subset of $\{1, 2, \ldots, n\}$ uniformly at random. Give a probability space to model this experiment and compute the probability of the following two events:

a) "the subset has cardinality 1".

b) "the subset contains 1"

2) We throw a dice with six faces which is not a fair dice, such that the following condition (H) is satisfied:

(H) : "the probability of falling on a face is proportional to its value."

Give a probability space $(\Omega, \mathbb{P})$ to model this experiment, write what condition (H) means using $\mathbb{P}$ and compute the probability that the dice falls on 6.

Solution of exercise 2.

1) We take $\Omega = \mathcal{P}(\{1, 2, \ldots, n\})$ (in particular, events are sets of subsets!) and $\mathbb{P}$ to be the uniform probability on Ω.

a) We compute the probability of the event E "the subset has cardinality 1", which is $E = \{\{1\}, \{2\}, \ldots, \{n\}\}$. Since $\#E = n$,

$$\mathbb{P}(E) = \frac{n}{2^n}.$$

b) We compute the probability of the event F "the subset contains 1". There are 2^{n-1} subsets of $\{1,2,\ldots,n\}$ containing 1 (since such subsets are in bijection with subsets of $\{2,3,\ldots,n\}$). Hence $\#F = 2^{n-1}$, so

$$\mathbb{P}(F) = \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

2) We take $\Omega = \{1,2,3,4,5,6\}$ and $\mathbb{P}$ is a probability on Ω such that there exists a constant $c > 0$ such that $\mathbb{P}(\{i\}) = c \times i$ for every $1 \le i \le 6$.

Side remark. By abuse of notation, if $(\Omega, \mathbb{P})$ is a finite probability space and $\omega \in \Omega$, we sometimes write $\mathbb{P}(\omega)$ instead of $\mathbb{P}(\{\omega\})$ to simplify notation. For example here, we could write $\mathbb{P}(i)$ instead of $\mathbb{P}(\{i\})$ (which is an abuse of notation, because by definition $\mathbb{P}$ is defined on $\mathcal{P}(\Omega)$). However, when elements of Ω are sets, this can cause some confusion...

To find c, we use the fact that $\mathbb{P}(\Omega) = 1$ and the fact that the probability of a disjoint union of events is the sum of their probabilities

$$1 = \mathbb{P}(\Omega) = \mathbb{P}\left(\cup_{i=1}^{6} \{i\}\right) = \sum_{i=1}^{6} \mathbb{P}(\{i\}) = \sum_{i=1}^{6} ci = 21c.$$

Hence $c = \frac{1}{21}$ and the probability of getting a 6 is $\frac{6}{21} = \frac{2}{7}$.

□

Exercise 3. Let $n \ge 1$ be an integer. The goal of this exercise is to study partitions of the set $\{1,2,\ldots,n\}$. By definition, a partition of $\{1,2,\ldots,n\}$ is a set of nonempty subsets of $\{1,2,\ldots,n\}$ which are pairwise disjoint and whose union is $\{1,2,\ldots,n\}$. We say that a partition is a k-partition if it has cardinality k.

For example, $\{\{1,8\},\{2,3,4,5,6,9\},\{7\}\}$ is a 3-partition of $\{1,2,\ldots,n\}$.

Denote by $B_{n,k}$ the total number of k-partitions of $\{1,2,\ldots,n\}$ and denote by B_n the total number of partitions of $\{1,2,\ldots,n\}$.

1) Write the set of all partitions of $\{1,2,3\}$.

2) Give the values of B_1, B_2, B_3.

3) What is the value of $B_{n,n-1}$?

4) What is the value of $B_{n,2}$?

5) Fix an integer $1 \le k \le n$.

a) Let $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$ be an onto map. For $1 \le i \le k$, set $A_i = f^{-1}(\{i\})$. Show that $\{A_1, A_2, \ldots, A_k\}$ is a k-partition of $\{1,2,\ldots,n\}$.

b) Let $S_{n,k}$ be the number of onto maps from $\{1,2,\ldots,n\}$ to $\{1,\ldots,k\}$. Show that $S_{n,k} = k! \times B_{n,k}$.

6) Set $B_0 = 1$. Show that for $n \ge 2$, $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$.

Remark. This formula gives a recursive way to compute the value of B_n. There is no simple expression for B_n.

Solution of exercise 3. 1) The partitions of $\{1,2,3\}$ are $\{\{1,2,3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$, $\{\{1\},$

$\{2\}, \{3\}\}$. Therefore the set of all partitions of $\{1,2,3\}$ is

$$\Big\{\big\{\{1,2,3\}\big\},\big\{\{1\},\{2,3\}\big\},\big\{\{2\},\{1,3\}\big\},\big\{\{3\},\{1,2\}\big\},\big\{\{1\},\{2\},\{3\}\big\}\Big\}.$$

2) There is only one partition of $\{1\}$ which is $\{\{1\}\}$, so $B_1 = 1$. There are two partitions of $\{1,2\}$ which are $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$, so $B_2 = 2$. By the previous question, $B_3 = 5$.

3) One sees that a $n-1$ partition of $\{1,2,\dots,n\}$ is made of $n-2$ sets having one element and on set having two elements. Therefore a $n-1$ partition of $\{1,2,\dots,n\}$ is characterized by the choice of 2 elements among n. Hence $B_{n,n-1} = \binom{n}{2}$.

4) A 2-partition of $\{1,2,\dots,n\}$ is characterized by the elements which are in the same set as 1. For every element $2,3,\dots,n$ we can either put it in the same set as 1 or not, which gives in total 2^{n-1} possibilities. However, we have to rule out the possibility that everyone is in the same set as 1 (recall that the subsets of the partition have to be nonempty). Therefore $B_{n,2} = 2^{n-1} - 1$.

5)

a) First, as f is onto, for every $1 \leq i \leq n$ we have that $A_i \neq \emptyset$.

Now we show that the sets are pairwise disjoint. Fix $1 \leq i,j \leq n$ with $i \neq j$. We argue by contradiction and assume that $A_i \neq A_j \neq \varnothing$. Take $x \in A_i \cap A_j$. Then, since $x \in A_i$, $f(x) = i$. Since $x \in A_j$, $f(x) = j$. But $i \neq j$, so this is a contradiction.

Finally we show that $A_1 \cup A_2 \cup \cdots \cup A_n = \{1,2,\dots,n\}$ by double inclusion. We clearly have the inclusion $\subset$. For the other inclusion, take $i \in \{1,2,\dots,n\}$. By definition, $i \in f^{-1}(\{f(i)\})$, so $i \in A_{f(i)}$. Hence $i \in A_1 \cup A_2 \cup \cdots \cup A_n$. This completes the proof.

b) An onto map from $\{1,2,\dots,n\}$ may be constructed uniquely as follows: first form a k-partition of n, then say which one of the k sets is $f^{-1}(\{1\})$, then which one is $f^{-1}(\{2\})$, and so one. By the product rule, this construction can be performed in $B_{n,k} \times k!$ ways by the product rule, and the result follows.

6) The idea is to generalize the argument of question 4) and to decompose the set of all partitions of $\{1,2,\dots,n\}$ according to the size of the subset containing 1. More precisely, a partition of $\{1,2,\dots,n\}$ may be uniquely constructed as follows: first fix the number k of elements added to the subset containing 1 (in addition to 1, so that $0 \leq k \leq n-1$). Then choose the k elements added to the subset containing 1 ($\binom{n-1}{k}$ ways). Then choose a partition of the remaining $n-k-1$ elements (B_{n-k-1} ways). By the product rule, this implies that the number of partitions of $\{1,2,\dots,n\}$ such that the number of elements of the set containing 1 is $k+1$ is equal to $\binom{n-1}{k} B_{n-k-1}$. The desired formula follows by the sum rule. □

2 Homework exercise

Exercise 4. Let $n \geq 1$ be an integer. We want to choose a subset of $\{1,2,\dots,n\}$ at random in such a way that the following condition (C) is satisfied:

(C) "there exists a value $a > 0$ such that the probability of choosing a subset containing 1 is a and the probability of choosing a subset not containing 1 is $2a$".

1) Give a probability space $(\Omega, \mathbb{P})$ to model this experiment, and write what condition (C) means using $\mathbb{P}$.

2) Give a simple expression of a involving only n, and justify your answer.

Solution of exercise 4.

1) We take $\Omega = \mathcal{P}(\{1,2,\ldots,n\})$ and $\mathbb{P}$ to be a probability on Ω satisfying $\mathbb{P}(\{A\}) = a$ if $1 \in A$ and $\mathbb{P}(\{A\}) = 2a$ if $1 \notin A$.

2) Since $\mathbb{P}(\Omega = 1)$, write

$$1 = \sum_{A\subseteq\{1,2,\ldots,n\}} \mathbb{P}(\{A\}) = \sum_{A\subseteq\{1,2,\ldots,n\},1\in A} \mathbb{P}(\{A\}) + \sum_{A\subseteq\{1,2,\ldots,n\},1\notin A} \mathbb{P}(\{A\}) = \sum_{A\subseteq\{1,2,\ldots,n\},1\in A} a + \sum_{A\subseteq\{1,2,\ldots,n\},1\notin A} 2a.$$

Hence, since there are 2^{n-1} subsets containing 1 and 2^{n-1} subsets not containing 1, we get

$$1 = 2^{n-1} \cdot a + 2^{n-1} \cdot 2a,$$

so

$$a = \frac{1}{3 \cdot 2^{n-1}}.$$

□

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 13.

Exercise 5. For $n \geq 1$, we call a *path* of length $2n$ any sequence $y_0, y_1, y_2, \ldots, y_{2n}$ of integers such that $y_0 = 0$ and for every $1 \leq i \leq 2n$, $y_i - y_{i-1} \in \{-1, +1\}$.

1) How many paths of length $2n$ are there ?

2) A path is called a *bridge* if $y_{2n} = 0$. How many bridges of length $2n$ are there?

Solution of exercise 5. 1) For every y_k, with $1 \leq k \leq 2n$, there's two choices, either $y_k = y_{k-1} + 1$ or $y_k = y_{k-1} + -1$. Thus by the product rule, the number of paths of length $2n$ is equal to $\underbrace{2 \times 2 \times \ldots \times 2}_{2n \text{ times}} = 2^{2n}$.

2) Observe that a path can be encoded by the set of its "+1" steps. For a bridge, there are necessarily n such steps, and a bridge is uniquely defined by the positions of its +1 steps. In other words, there is a bijection

$$\{\text{Bridges of length } 2n\} \leftrightarrow \{\text{Subsets of size } n \text{ included in } \{1,2,\ldots,2n\}\}.$$

As a consequence, $\#\{\text{Bridges of length } 2n\} = \binom{2n}{n}$. □

Exercise 6. Let $1 \leq p \leq n$ be integers. Let E be a set with n elements and A a subset of E with p elements.

1) How many subsets X of E such that $A \subset X$ are there?

2) If $p \leq m \leq n$, how many subsets X of E such that $A \subset X$ are there?

3) How many couples (X, Y) of subsets of E such that $X \cap Y = A$ are there?

Solution of exercise 6.

1) As many as the number of subsets of $E \backslash A$ (which correspond to the elements we add to A to obtain X), that is 2^{n-p}.

2) As many as the number of subsets of $E \backslash A$ having $m-p$ elements, that is $\binom{n-p}{m-p}$.

3) Once we have chosen a subset X of E such that $A \subset X$ and $\text{Card}(X) = m$ ($\binom{n-p}{m-p}$ choices), we have to choose for Y a subset of $E \backslash A$ (2^{n-m} choices). The answer is therefore

$$\sum_{m=p}^{n} \binom{n-p}{m-p} 2^{n-m} = \sum_{k=0}^{n-p} \binom{n-p}{k} 2^{n-p-k} = (1+2)^{n-p} = 3^{n-p},$$

where we have used the Binomial theorem for the second equality. □

Exercise 7. Let $n \geq 2$ be an integer and let us consider a deck of n cards numbered from 1 to n.

1) In how many ways is it possible to shuffle the deck so that the card with number 1 is further in the deck than the card 2?

2) In how many ways is it possible to shuffle the deck so that the cards with numbers 1 and 2 are neighbours ?

Solution of exercise 7. We may view a shuffling of the deck as a permutation $\sigma \in S_n$.

1. We want to count the number of elements of the set $A = \{\sigma \in S_n,\ \sigma(1) > \sigma(2)\}$. To this end, we partition A according to the value of $k = \sigma(1)$. Once this $k \geq 2$ has been chose, we have to chose:

   - the value of $\sigma(2)$, with $k-1$ choices (positive integers less that k) ,
   - then a bijection between $\{3, 4, \ldots, n\}$ and $\{1, 2, \ldots, n\} \setminus \{\sigma(1), \sigma(2)\}$, with $(n-2)!$ choices.

   Therefore
   $$\text{Card}(A) = \sum_{k=2}^{n} (k-1)(n-2)! = (n-2)! \frac{(n-1)n}{2} = \frac{n!}{2}.$$

2. We want to count the number of elements of the set $B = \{\sigma \in S_n,\ |\sigma(1) - \sigma(2)| = 1\}$. To this end, we partition B according to the value of $k = \sigma(1)$ and then according to the value of $\sigma(2)$ (only one choice if $k = 1$ or $k = n$, two choices otherwise); it then remains to choose a bijection between $\{3, 4, \ldots, n\}$ and $\{1, 2, \ldots, n\} \setminus \{\sigma(1), \sigma(2)\}$. Therefore
   $$\text{Card}(B) = (n-2)! + \sum_{k=2}^{n-1} 2(n-2)! + (n-2)! = (n-2)!(2n-2) = 2(n-1)!.$$

□

Exercise 8. Show that for every $n \geq 1$, $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

Solution of exercise 8. We establish this identity by a combinatorial argument, by showing equivalently that

$$\sum_{k=0}^{n} k\binom{n}{k}\binom{n}{n-k} = n\binom{2n-1}{n-1}.$$

To this end, imagine a group of n men and n women. We count in two ways the number of possibilities of choosing a group of n people and choosing a female leader among this group.

First way. We partition the total number of possibilities according to the number $0 \leq k \leq n$ of women in the group. There are $\binom{n}{k}$ ways of choosing k women, then $\binom{n}{n-k}$ ways of choosing $n-k$ men, and then k ways of choosing a female leader, which gives $k\binom{n}{k}\binom{n}{n-k}$ possibilities by the product rule. By the sum rule, the total number of possibilities is $\sum_{k=0}^{n} k\binom{n}{k}\binom{n}{n-k}$.

Second way. We first choose a female leader (n choices) and then complete the groupe by choosing $n-1$ people among $2n-1$. This gives $n\binom{2n-1}{n-1}$ possibilities. □

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 13.

Exercise 9. 71 mathematicians are standing in a line, wearing a black or white hat. Each mathematician can ONLY see the color of the hats of the people in front of them. So the first person sees no hats, the last sees 70. The mathematicians are allowed to talk to each other and decide upon a strategy, for a government rep is coming to cut off funding. Each person can only say "black" or "white." If you correctly say the color of the hat you're wearing, your funding is continued and you live. If you're wrong, you lose your funding, and you may as well be dead.

How many mathematicians can you guarantee will keep their funding?

(You are not allowed to use "tricks," say a person delays one second before answering means A, two seconds means B, ... You have to answer IMMEDIATELY what color hat you're wearing.)

Solution of exercise 9. It is possible to guarantee the funding of 70 mathematicians. The strategy is the following: first the mathematician A who sees 70 hats says "white" if she sees an odd number of white hats, and "black" if she sees an even number of white hats. This allows the mathematician B who sees 69 hats to deduce the colour of her hat (indeed, if A said "white" and B still sees an odd number of white hats, then B has a black hat, and if B sees an even number of white hats, then B has a white hat, and B similarly finds the colour of her hat if A says "black"). The mathematician C who sees 68 hats can then deduce the colour of her hat, and so one until the last one. □