Week 12: Finite probability spaces

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"The theory of probability, as a mathematical discipline, can and should be developed from axioms in exactly the same way as Geometry and Algebra. This means that after we have defined the elements to be studied and their basic relations, and have stated the axioms by which these relations are to be governed, all further exposition must be based exclusively on these axioms, independent of the usual concrete meaning of these elements and their relations. "

Andreï Kolmogorov (1956).

Kolmogorov explains that the mathematical theory of probability is developed independent of all the real world restrictions, only being subject to the constraints of logic. *The misconception to think that a mathematical discipline automatically deals with something real has led to unrealistic models.*

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Fix integers $1 \leq r \leq n$. We put r balls (numbered from 1 to r) into n urns (numbered from 1 to n), uniformly at random.

1) Construct a finite probability space to model this experiment.
2) Find the probability that every urn has at most one ball.
3) Find the probability that there is an urn having at least two balls.

Exercise 2. Let A, B, C be events of a finite probability space. Using set operations, write the following events:

(a) A is not realized
(b) None of the events A, B nor C are realized.
(c) Only one of the events A, B or C is realized.
(d) At least two of the events A, B, C are realized.
(e) No more that two of the events A, B, C are realized

Exercise 3. Consider a parking lot having 8 consecutive slots (meaning that the slots are one after the other). A blue car and a red car have parked uniformly at random.

1) Construct a finite probability space to model this experiment.
2) What is the probability that the first slot has been taken by a car?
3) What is the probability that the two cars have parked next to each other?

Exercise 4. Let $(\Omega, \mathbb{P})$ be a finite probability space.

1) If A, B are events show that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Hint: write $A \cup B = (A \backslash A \cap B) \cup (A \cap B) \cup (B \backslash A \cap B)$.

2) Let $n \geq 2$ be an integer. Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events. Show that $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mathbb{P}(A_k)$.

3) *(Application)* Let $n \geq 2$ be an integer.

a) Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events such that $\mathbb{P}(A_k) = 0$ for every $1 \leq k \leq n$. Show that $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 0$.

b) Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events such that $\mathbb{P}(A_k) = 1$ for every $1 \leq k \leq n$. Show that $\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) = 1$.

2 Homework exercise

You have to individually hand in the written solution of the next exercise to your TA on Monday, January 6th.

Exercise 5. Five people each throw a fair dice (a dice is fair when the probability of falling on its different faces is the same). Among the five people, three people have one dice with 6 faces from 1 to 6 and two people have one dice with 4 faces from 1 to 4.

1) Construct a finite probability space to model this experiment.

2) Compute the probabilities of the following events:

(a) everyone gets 4; (b) everyone gets 5 (c) all the numbers are different;

(d) at least two people obtain the same number; (e) one of the dices having 6 faces gives the same number as one of the dices having 4 faces.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 12.

Exercise 6. Fix integers $1 \leq r \leq n$. We put r (indistinguishable) purple balls into n urns (numbered from 1 to n), uniformly at random.

1) Construct a finite probability space to model this experiment.

2) Find the probability that every urn has at most one ball.

Exercise 7. Let $A_1, \ldots, A_n$ be events of a finite probability space $(\Omega, \mathbb{P})$.

1) Show that $\mathbb{P}(A_1 \cap \cdots \cap A_n) \leq \min_{1 \leq i \leq n} \mathbb{P}(A_i)$.

2) Show that $\mathbb{P}(A_1 \cap \cdots \cap A_n) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - (n-1)$.

Exercise 8. Let $(\Omega, \mathbb{P})$ be a finite probability space, $n \geq 1$ an integer and $A_1, \ldots, A_n$ events. Set $[\![1, n]\!] = \{1, 2, \ldots, n\}$. Show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I \subseteq [\![1,n]\!] \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \mathbb{P}\left(\bigcap_{i \in I} A_i\right).$$

Exercise 9. Let $(\Omega, \mathbb{P})$ be a finite probability space and fix two events A, B. Show that $|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \frac{1}{4}$.

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 12.

Exercise 10. A colony of n vampires lives in the Carpathian mountains. The Countess Dracula wishes to estimate the number n of vampires (Dracula is not considered as being part of the colony). To do so, one night, she captures ten of them at random, bits their ears and releases them. The next night, she captures again 10 vampires at random. It turns out that 3 of them have their ears bitten.

Denote by p_n the probability that 3 vampires have their ears bitten when one chooses 10 of them uniformly at random in a population of n vampires with 10 having their ears bitten. For what integer n is the quantity p_n maximal?

Hint. You may simplify the quantity $\frac{p_n}{p_{n+1}}$.