Week 1, September 23: sets

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. What is the cardinality of the set $\{1,\{1,\{2,3\}\},\emptyset\}$?

Exercise 2. Which of the following statements are true? Justify your answer.

a) $2 \in \{1,2,3\}$ *b*) $\{2\} \subseteq \{1,2,3\}$ *c*) $\{2\} \in \{1,2,3\}$ *d*) $\{2,2,3,3,3\} \subseteq \{1,2,3\}$ *e*) $\{2\} \subseteq \{\{1\},\{2\}\}$
f) $2 \subseteq \{1,2,3\}$ *g*) $\{2\} \in \{\{1\},\{2\}\}$ *h*) $\emptyset \in \{\emptyset\}$ *i*) $\emptyset \subseteq \{\emptyset\}$ *j*) $\{2\} \in 2$

Exercise 3. If A and B are two sets, by definition, the *complement of A in B* is the set of all elements in B which are not in A. We write $B\backslash A$ and read also "B without A" (in other words, $B\backslash A = \{x \in B : x \notin A\}$). When one works with subsets of a "big" set E, one often writes $\overline{A}$ for the complement a set A in E.

a) What is the complement of $\{2,3\}$ in $\{\{1\},2,\emptyset\}$?

b) What is the complement of $\{\{1\},2,4\}$ in $\{\{1\},2,\emptyset\}$?

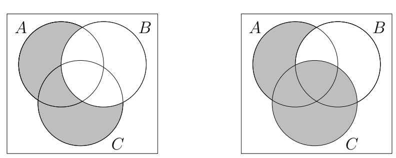
c) What is the complement of $\{1,2,3\}$ in $\{1,2\}$?

d) What is the complement of 2 in $\{\{1\},2,\emptyset\}$?

Exercise 4. If A and B are sets, the union of A and B, denoted by $A\cup B$, is by definition the set of all elements which are in A or in B (or in both), that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$. The intersection of A and B, denoted by $A\cap B$, is by definition the set of all elements which are in A and in B, that is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

1) If $A = \{1,\{2\},\emptyset\}$ and $B = \{2,\{2\}\}$, what are $A\cup B$ and $A \cap B$?

2) In the following two examples (where A,B,C are subsets of a set E), describe the elements of the gray region using the symbols $\cap$ (union), $\cup$ (intersection) and $^{-}$ (complement in E).



Exercise 5. Let A,B,C be sets. Show that $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

Exercise 6. Let A,B be subsets of a set E. Prove that $\overline{A\cup B} = \overline{A} \cap \overline{B}$.

Exercise 7. Let A,B be two subsets of E. What can we say if $A\cap B = A\cup B$? Prove it!

Exercise 8. Let A be any set. The *power set* of A, denoted by $\mathcal{P}(A)$ is the set of all the subsets of A.

1. Give the power set of $\{1,2,3\}$.

2. Let A be a set with $n \geq 1$ elements. What is the cardinality of the power set of A?

2 Homework exercises

You have to individually hand in the written solutions of the next two exercises to your TA on September 30.

Exercise 9. Let A, B, C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 10. If A and B are two sets, we define $A \Delta B = (A \cup B) \backslash (A \cap B)$, where we recall that $Y \backslash X$ is the set of all elements of Y which do not belong to X.

1) If A and B are the intervals $A = [-2, 3]$ and $B = (0, 5]$ (meaning that 0 is excluded), what is $A \Delta B$? Justify your answer.

2) Let A and B be two subsets of a set E. Show that $A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$, where $\overline{X}$ denotes the complement of a set X in E.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 1.

Exercise 11. By definition, a couple (a, b) is an ordered collection of the two objects a and b. For example $(1, 2)$, $(2, 1)$ and $(1, 1)$ are three different couples. The *cartesian product* $A \times B$ of two sets A, B is the set defined by $A \times B = \{(a, b) : a \in A, b \in B\}$.

1. When $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, write $A \times B$ and $B \times A$ by listing its elements.
2. If A and B are finite sets, what is the cardinality of $A \times B$?
3. Is the equality $A \times (B \times C) = (A \times B) \times C$ always true?

Exercise 12. If A is a set, recall that $\mathcal{P}(A)$ denotes the set of all subsets of A. What are the elements of $\mathcal{P}(\mathcal{P}(\{\emptyset\}))$?

Exercise 13. Let $m, n \geq 1$ be integers. A pawn, called (m, n)-capybara, moves on a board having infinitely many horizontal lines (infinitely many upwards and infinitely many downwards) and infinitely many vertical columns (infinitely many to the left and infinitely many to the right). At each step, the capybara moves m squares in one direction (horizontal or vertical), then n squares in the perpendicular direction. For example, a chess horse is a $(2, 1)$-capybara. For which values of m and n is it possible to colour the squares of the board in blue or in red so that the capybara sees a different colour each time no matter how it moves?

Three capybaras

4 Fun exercise (optional)

The "solution" of this exercise will be available on the course webpage at the end of week 1.

Exercise 14. What do you think of the following definition?

Let n be the smallest number that cannot be described in English using 20 words or fewer.