Week 1, September 23: sets

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. What is the cardinality of the set $\{1,\{1,\{2,3\}\},\emptyset\}$?

Solution of exercise 1. The answer is 3. Indeed, the elements of this set are 1, $\{1,\{2,3\}\}$ and $\emptyset$. □

Exercise 2. Which of the following statements are true? Justify your answer.

a) $2 \in \{1,2,3\}$ *b*) $\{2\} \subseteq \{1,2,3\}$ *c*) $\{2\} \in \{1,2,3\}$ *d*) $\{2,2,3,3,3\} \subseteq \{1,2,3\}$ *e*) $\{2\} \subseteq \{\{1\},\{2\}\}$
f) $2 \subseteq \{1,2,3\}$ *g*) $\{2\} \in \{\{1\},\{2\}\}$ *h*) $\emptyset \in \{\emptyset\}$ *i*) $\emptyset \subseteq \{\emptyset\}$ *j*) $\{2\} \in 2$

Solution of exercise 2.

a) True: 2 is indeed an element of $\{1,2,3\}$

b) True: all the elements of the set $\{2\}$ are elements of the set $\{1,2,3\}$.

c) False: $\{2\}$ is not an element of $\{1,2,3\}$ (indeed, $\{2\}$ is a set, but the elements of $\{1,2,3\}$ are integers).

d) True: all the elements of the set $\{2,2,3,3,3\}$ are elements of the set $\{1,2,3\}$.

e) False: the element 2 of the set $\{2\}$ is not an element of the set $\{\{1\},\{2\}\}$.

f) This statement does even not make any sense: the statement $A \subseteq B$ only makes sense when A and B are sets.

g) True: $\{2\}$ is indeed an element of the set $\{\{1\},\{2\}\}$.

h) True: $\emptyset$ is an element of the set $\{\emptyset\}$

(i) True: by convention, we always have $\emptyset \subseteq A$ for any set A.

(j) This statement does even not make any sense: the statement $x \in A$ only makes sense when A is a set.

□

Exercise 3. If A and B are two sets, by definition, the *complement of A in B* is the set of all elements in B which are not in A. We write $B \backslash A$ and read also "B without A" (in other words, $B \backslash A = \{x \in B : x \notin A\}$). When one works with subsets of a "big" set E, one often writes $\overline{A}$ for the complement a set A in E.

a) What is the complement of $\{2,3\}$ in $\{\{1\},2,\emptyset\}$?

b) What is the complement of $\{\{1\},2,4\}$ in $\{\{1\},2,\emptyset\}$?

c) What is the complement of $\{1,2,3\}$ in $\{1,2\}$?

d) What is the complement of 2 in $\{\{1\},2,\emptyset\}$?

Solution of exercise 3.

a) The set of elements of $\{\{1\},2,\emptyset\}$ which are not elements of $\{2,3\}$ is $\{\{1\},\emptyset\}$.

b) The set of elements of $\{\{1\},2,\emptyset\}$ which are not elements of $\{\{1\},2,4\}$ is $\{\emptyset\}$.

c) There are no elements of $\{1,2\}$ which are not elements of $\{1,2,3\}$, so the answer is $\emptyset$.

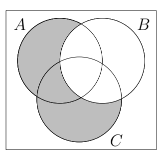
d) This statement does not make any sense: one may speak of the complement of A in B only when A and B are sets. Here, 2 is not a set.

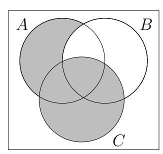
□

Exercise 4. If A and B are sets, the union of A and B, denoted by $A \cup B$, is by definition the set of all elements which are in A or in B (or in both), that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$. The intersection of A and B, denoted by $A \cap B$, is by definition the set of all elements which are in A and in B, that is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

1) If $A = \{1,\{2\},\emptyset\}$ and $B = \{2,\{2\}\}$, what are $A \cup B$ and $A \cap B$?

2) In the following two examples (where A,B,C are subsets of a set E), describe the elements of the gray region using the symbols $\cap$ (union), $\cup$ (intersection) and $^-$ (complement in E).





Solution of exercise 4. 1) We have $A \cup B = \{1,2,\{2\},\emptyset\}$ and $A \cap B = \{\{2\}\}$.

2)

$$(A \cup C) \cap \overline{B}, \qquad C \cup \left(A \cap \overline{B}\right)$$

□

Exercise 5. Let A,B,C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution of exercise 5. We argue by double inclusion.

First, take $x \in A \cup (B \cap C)$.

Case 1: $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$.

Case 2: $x \in B \cap C$. Then $x \in B$ and $x \in C$. Therefore $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$.

We conclude that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now, take $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$.

Case 1: $x \in A$. Then $x \in A \cup (B \cap C)$.

Case 2: $x \notin A$. Then we must have $x \in B$ and $x \in C$, which implies $x \in B \cap C$. Therefore $x \in A \cup (B \cap C)$. We conclude that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

This completes the proof. □

Exercise 6. Let A, B be subsets of a set E. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Solution of exercise 6. We argue by double inclusion.

First take $x \in \overline{A \cup B}$. This means that $x \notin A \cup B$. Hence $x \notin A$ (otherwise we would have $x \in A \cup B$ which is not the case) and $x \notin B$ (otherwise we would have $x \in A \cup B$ which is not the case). Hence $x \in \overline{A}$ and $x \in \overline{B}$. Hence $x \in \overline{A} \cap \overline{B}$. Therefore $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Second take $x \in \overline{A} \cap \overline{B}$. Hence $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$ (indeed, if $x \in A \cup B$, we would have $x \in A$ or $x \in B$, which is not the case). Hence $x \in \overline{A \cup B}$. Therefore $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. □

Exercise 7. Let A, B be two subsets of E. What can we say if $A \cap B = A \cup B$? Prove it!

Solution of exercise 7. Let us prove that $A = B$ by arguing by double inclusion.

First,

$$A \subseteq A \cup B = A \cap B \subseteq B.$$

Similarly,

$$B \subseteq A \cup B = A \cap B \subseteq A.$$

We have proved that $A \subseteq B$ and $B \subseteq A$, which implies that $A = B$. □

Exercise 8. Let A be any set. The *power set* of A, denoted by $\mathcal{P}(A)$ is the set of all the subsets of A.

1. Give the power set of $\{1, 2, 3\}$.

2. Let A be a set with $n \geq 1$ elements. What is the cardinality of the power set of A?

Solution of exercise 8.

1. The power set of $\{1, 2, 3\}$ is $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, which has 8 elements.

2. Assume that $A = \{a_1, a_2, \ldots, a_n\}$. To each subset B of A, we can associate a list of length n using the words "yes" or "no". The k-th is "yes" if $a_k \in B$, "no" otherwise. This correspondence shows that $|\mathcal{P}(A)|$ is the same as the number of length-n "yes"/"no" lists. There are 2^n such lists.

□

2 Homework exercises

You have to individually hand in the written solutions of the next two exercises to your TA on September 30.

Exercise 9. Let A, B, C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution of exercise 9. We argue by double inclusion.

Take $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

Case 1: $x \in B$. Then $x \in A \cap B$, so that $x \in (A \cap B) \cup (A \cap C)$.

Case 2: $x \in C$. Then $x \in A \cap C$, so that $x \in (A \cap B) \cup (A \cap C)$.

We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now take $x \in (A \cap B) \cup (A \cap C)$.

Case 1: $x \in A \cap B$. Then $x \in A$ and $x \in B \cup C$, so that $x \in A \cap (B \cup C)$.

Case 2: $x \in A \cap C$. Then $x \in A$ and $x \in C \cup B$, so that $x \in A \cap (B \cup C)$.

We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof. □

Exercise 10. If A and B are two sets, we define $A \Delta B = (A \cup B) \backslash (A \cap B)$, where we recall that $Y \backslash X$ is the set of all elements of Y which do not belong to X.

1) If A and B are the intervals $A = [-2, 3]$ and $B = (0, 5]$ (meaning that 0 is excluded), what is $A \Delta B$? Justify your answer.

2) Let A and B be two subsets of a set E. Show that $A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$, where $\overline{X}$ denotes the complement of a set X in E.

Solution of exercise 10.

1) We have $A \cup B = [-2, 5]$ and $A \cap B = (0, 3]$. Therefore $A \Delta B = [-2, 0] \cup (3, 5]$.

2) We argue by double inclusion to show that $(A \cup B) \backslash (A \cap B) = (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

First take $x \in (A \cup B) \backslash (A \cap B)$. Then $x \in A \cup B$.

Case 1: $x \in A$. Then $x \notin B$ (because $x \notin A \cap B$). Therefore $x \in A \cap \overline{B}$, so that $x \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

Case 2: $x \in B$. Then, similarly, $x \notin A$ (because $x \notin A \cap B$). Therefore $x \in \overline{A} \cap B$, so that $x \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

Next, take $x \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

Case 1: $x \in A \cap \overline{B}$. Then $x \in A$, so that $x \in A \cup B$. Also, $x \notin A \cap B$ (since $x \notin B$). We conclude that $x \in (A \cup B) \backslash (A \cap B)$.

Case 2: $x \in \overline{A} \cap B$. Then $x \in B$, so that $x \in A \cup B$. Also, $x \notin A \cap B$ (since $x \notin A$). We conclude that $x \in (A \cup B) \backslash (A \cap B)$.

This completes the proof. □

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 1.

Exercise 11. By definition, a couple (a, b) is an ordered collection of the two objects a and b. For example $(1, 2)$, $(2, 1)$ and $(1, 1)$ are three different couples. The *cartesian product* $A \times B$ of two sets A, B is the set defined by $A \times B = \{(a, b) : a \in A, b \in B\}$.

1. When $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, write $A \times B$ and $B \times A$ by listing its elements.
2. If A and B are finite sets, what is the cardinality of $A \times B$?
3. Is the equality $A \times (B \times C) = (A \times B) \times C$ always true?

Solution of exercise 11.

1. We have

$$A \times B = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}, \quad B \times A = \{(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\}.$$

   In particular, note that $A \times B \neq B \times A$.

2. The cardinal of $A \times B$ is $\mathrm{card}(A) \times \mathrm{card}(B)$. Indeed, choosing an element of $A \times B$ amounts to choosing an element of A (for the first position) and then an element of B (for the second position), which gives $\mathrm{card}(A) \times \mathrm{card}(B)$ choices overall.

3. The formula is not correct. For instance, taking $A = \{1\}$, $B = \{2\}$ and $C = \{3\}$.

$$\{1\} \times \Big(\{2\} \times \{3\}\Big) = \Big\{(1,(2,3))\Big\}, \qquad \Big(\{1\} \times \{2\}\Big) \times \{3\} = \Big\{((1,2),3)\Big\}.$$

   which are not equal, since $(1,(2,3)) \neq ((1,2),3)$.

□

Exercise 12. If A is a set, recall that $\mathcal{P}(A)$ denotes the set of all subsets of A. What are the elements of $\mathcal{P}(\mathcal{P}(\{\emptyset\}))$?

Solution of exercise 12. We have $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, so that

$$\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}.$$

□

Exercise 13. Let $m, n \geq 1$ be integers. A pawn, called (m,n)-capybara, moves on a board having infinitely many horizontal lines (infinitely many upwards and infinitely many downwards) and infinitely many vertical columns (infinitely many to the left and infinitely many to the right). At each step, the capybara moves m squares in one direction (horizontal or vertical), then n squares in the perpendicular direction. For example, a chess horse is a $(2,1)$-capybara. For which values of m and n is it possible to colour the squares of the board in blue or in red so that the capybara sees a different colour each time no matter how it moves?

Three capybaras

Solution of exercise 13. We shall show that it is always possible. The idea is to treat different cases according to the parity of m and n.

a) Let us start with the $(2,1)$-capybara, who is just a chess horse. In this case, one may colour the board as the left-most part of Figure 1). The reason is that two squares of this grid have the same colour if and only if one may go from one another by doing

- either an even number of horizontal and vertical squares,
- or an odd number of squares in both direction (horizontal and vertical).

Since a $(2,1)$-capybara moves an odd number of squares in a direction and an even number in the other direction, the colour of the square containing the capybara changes each time the capybara moves.

More generally, the same reasoning shows that this colouring works when m and n have different parity.

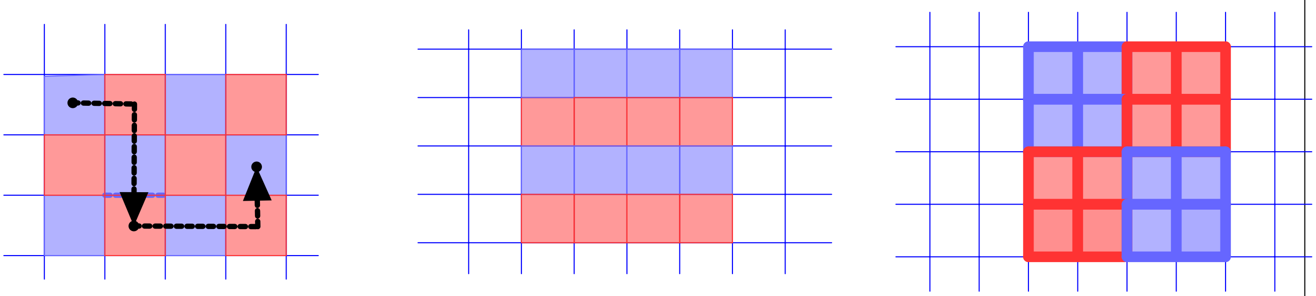


Figure 1: From left to right: a) m and n have different parity, b) m and n are odd, c) $\gcd(m,n) \neq 1$.

b) Now assume that m and n are odd. Then at each step, the capybara will always move an odd number of squares vertically. This gives the idea to colour the board as the middle image of Figure 1. One indeed sees that in this colouring, two squares have the same colour if and only if they are on the same horizontal line or if their horizontal lines are separated by an even number of squares.

c) The case where both m and n are even is treated by reduction to the previous cases. Indeed, let $d = \gcd(m,n)$ be the greatest common divisor of m and n. We can then write $n = n'd$ and $m = m'd$, where n' and m' are two integers which are coprime. In particular, m' and n' cannot be both even. One then colours the board by using large $d \times d$ squares of the same colour corresponding to the solution of the (n',m')-capybara. See the right-most part of Figure 1 for an illustration of the case $d = 2$ and where n' and m' have different parity. □

4 Fun exercise (optional)

The "solution" of this exercise will be available on the course webpage at the end of week 1.

Exercise 14. What do you think of the following definition?

Let n be the smallest number that cannot be described in English using 20 words or fewer.

Solution of exercise 14. Well, on one hand this n seems to exist since there are only a finite number of sentences with less than 20 words. But the actual text of this exercise is a description of this particular n in only 17 words!

You can google "Berry paradox" for more on this issue. □