Week 8 (Midterm exam): Tuesday, November 12th, 8am-10am

Very important:

- Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).
- Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Advice. Use draft paper before writing your answers in the final form. Reread your work. Do not forget that what is graded is what is written, not what is in your head.

Part 1

Exercise 1.

1) Give an example of a function which is not onto (and explain why it is not onto) and an example of a function which is not one-to-one (and explain why it is not one-to-one).

2) Show that if $f : X \to Y$ and $g : Y \to Z$ are bijections, then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(You may use without proof the fact that $g^{-1}(g(b)) = b$ for every $b \in Y$ and that $f^{-1}(f(a)) = a$ for every $a \in X$.)

Part 2

Exercise 2. Let P, Q, R be three mathematical assertions.

1) Are the two assertions $P \implies (Q \wedge R)$ and $(P \implies Q) \wedge (P \implies R)$ logically equivalent? Justify your answer.

2) Are the two assertions $(Q \wedge R) \implies P$ and $(Q \implies P) \wedge (R \implies P)$ logically equivalent? Justify your answer.

Exercise 3. Let A, B, C be three sets. Show that $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$.

Part 3

Exercise 4. Among the following assertions, which ones are true? Justify your answers.

a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, xy > 0$

b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > 0$

c) $\exists x \in \mathbb{R}, \forall y \in R, xy > 0$

d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 > x$

Exercise 5. Let E and F be two sets and let $f : E \to F$ be a function. Show that

$$f \text{ is one-to-one} \qquad \Longleftrightarrow \qquad \forall A, B \subseteq E, \quad f(A \cap B) = f(A) \cap f(B).$$

You may use without proof that if U, V are subsets of E such that $U \subseteq V$, then $f(U) \subseteq f(V)$.

Part 4

Exercise 6. Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

1) Show that if $g \circ f$ is one-to-one and f is onto, then g is one-to-one.

2) Show that if $g \circ f$ is onto and g one-to-one, then f is onto.

3) Is it always true that if $g \circ f$ is bijective, then g or f is bijective? Justify your answer.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let E be a set.

1) Show that E has infinitely many elements if and only if for every function $f : E \to E$ there exists a set $A \subseteq E$ with $A \neq \emptyset$, $A \neq E$ such that $f(A) \subseteq A$.

2) Is it true that E has infinitely many elements if and only if for every function $f : E \to E$ there exists a set $A \subseteq E$ with $A \neq \emptyset$, $A \neq E$ such that $f(A) = A$?

Exercise 8. Let E be a set. Recall that $\mathcal{P}(E)$ denotes the set of all its subsets. Let $F : \mathcal{P}(E) \to \mathcal{P}(E)$ be a function such that for every $A, B \in \mathcal{P}(E)$, $A \subseteq B \implies F(A) \subseteq F(B)$. Show that there exists $M \in \mathcal{P}(E)$ such that $F(M) = M$.

Hint. You may introduce the set $\mathcal{S} = \{A \in \mathcal{P}(E) : F(A) \subseteq A\}$ and define $M = \bigcap_{A \in \mathcal{S}} A$.

Exercise 9. What does the following image prove?