Week 9 (Midterm exam): Tuesday, December 4th, 8am-10am

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1.

1) Show that if $f : X \to Y$ and $g : Y \to Z$ are two one-to-one functions, then $g \circ f$ is one-to-one.

2) Show that if $f : X \to Y$ and $g : Y \to Z$ are two onto functions, then $g \circ f$ is onto.

Part 2

Exercise 2. Let A, B, C be three sets. Show that $A \cap C = A \cup B \iff B \subset A$ and $A \subset C$.

Exercise 3. Let E be a set and $f : E \to E$ a function such that $f(f(x)) = f(x)$ for every $x \in E$.

1) Show that f is one-to-one if and only if f is onto.

2) Give an example of a set E and of a function $f : E \to E$ such that $f(f(x)) = f(x)$ for every $x \in E$ and such that f is a bijection (and explain why it is a bijection).

3) Give an example of a set E and of a function $f : E \to E$ such that $f(f(x)) = f(x)$ for every $x \in E$ and such that f is not a bijection (and explain why it is not a bijection).

Part 3

Exercise 4. Show that for every integer $n \geq 1$, $\sum_{k=1}^{n} k \times k! = (n+1)! - 1$.

Exercise 5. Professor B. goes to a shop to buy 3 different books about kittens and 2 different books about capybaras. In the shop there are 10 different books about kittens and 20 different books about capybaras.

Note: *in the following questions, please give numbers as answers (without binomial coefficients nor factorials) and justify your answers ("informal" proofs are allowed and recommended).*

1) In how many ways can Professor B. buy his books?

2) Once home, Professor B. forms a (vertical) pile of his books. How many different piles can he make?

3) Same question as 2), if the books about kittens have to be at the top of the pile and the books about capybaras at the bottom of the pile.

4) Same question as 2), if two books about capybaras cannot be one on the other.

Part 4

Exercise 6. This exercise studies a simplified model of a protein with 3 amino-acids. Let $n \geq 1$ be an integer. Let S_n be the set of all words with n letters formed by using the letters C, D, E and such that the following two conditions hold:

(∗) the words start with the letter C

(∗) there is never two times the same letter one after the other.

For example $S_3 = \{CDC, CDE, CED, CEC\}$.

Let C_n be the subset of S_n made of words finishing with the letter C, let D_n be the subset of S_n made of words finishing with the letter D and let E_n be the subset of S_n made of words finishing with the letter E. Finally set $c_n = \#C_n$, $d_n = \#D_n$, $e_n = \#E_n$.

Remark. *If at some point you do not manage to solve a question, you can write on your sheet of paper that you assume that it is true and use it to solve a next question if needed.*

1) Compute $c_1, d_1, e_1, c_2, d_2, e_2, c_3, d_3, e_3$.
2) What is the value of $\#S_n$? Justify your answer.
3) Show that $c_{n+1} = d_n + e_n$, $d_{n+1} = c_n + e_n$, $e_{n+1} = c_n + d_n$ for $n \geq 1$.
4) Show that $c_{n+2} = c_{n+1} + 2c_n$ for $n \geq 1$.
5) Show that $c_n = \frac{2^n}{6} + \frac{2}{3}(-1)^{n+1}$ for every $n \geq 1$ and find a similar simple formula for d_n and e_n.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Recall that a partition of an integer $n \geq 1$ into k parts is a sequence $(x_1, \ldots, x_k)$ of k positive integers such that $x_1 \geq x_2 \geq \cdots \geq x_k$ and $x_1 + x_2 + \cdots + x_k = n$. Show that the number of partitions of n into at most k parts is equal to the number of partitions of $n + k$ into exactly k parts.

Exercise 8. Let $n \geq 1$ be an integer. How many words of length n can be formed from the letters A,B,C if the letter A has to occur an even number of times?

Exercise 9. What does the following image represent?