Week 17 (Final exam): Friday, February 1st, 14:00-16:00 pm

Very important:

- Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).
- Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Remark. *If at some point you do not manage to solve a question, you can write on your sheet of paper that you assume that it is true and use it to solve a next question if needed.*

Part 1

Exercise 1.

1) Give the definition of a probability on a finite state space Ω.

2) Let $(\Omega, \mathbb{P})$ be a finite probability space. Let B be an event such that $\mathbb{P}(B) > 0$. Show that the function

$$\mathbb{P}_B \ : \ \begin{array}{ccc} \mathcal{P}(\Omega) & \to & [0,1] \\ A & \mapsto & \mathbb{P}(A|B) \end{array}$$

is a probability on Ω.

Exercise 2. Consider the following three assertions:

(A) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y > 0$, (B) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y > 0$, (C) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y > 0$.

For each one of the assertions, say if it is true or false (and justify your answer).

Part 2

Exercise 3. Let $n \geq 1$ be an integer.

1) Show that $\binom{2n+2}{n+1} = \left(4 - \frac{2}{n+1}\right)\binom{2n}{n}$.

2) Show that $\binom{2n}{n} < 4^n$.

Exercise 4. Let E be a set and let $f : E \to E$ be a function such that $f \circ f \circ f = f$.

1) Show that f is onto if and only if f is one-to-one.

2) Give an example of a set E and of a function $f : E \to E$ be a function such that $f \circ f \circ f = f$ which is neither onto, neither one-to-one (and explain why it is neither onto, neither one-to-one).

Part 3

Exercise 5. Let $n \geq 2$ be an integer. We consider an urn with n numbered balls from 1 to n. We choose at random one ball, put it back in the urn, and then choose at random another ball. We model this experiment with the state space $\Omega = \{1, 2, \ldots, n\}^2$ equipped with the uniform probability $\mathbb{P}$. Fix an integer $k \in \{1, 2, \ldots, n\}$.

1) Let A_k be the event "the maximum of the two balls is less than or equal to k". Compute $\mathbb{P}(A_k)$ and justify your answer.

2) Let B_k be the event "the maximum of the two balls is equal to k". Compute $\mathbb{P}(B_k)$ and justify your answer.

3) Let C_k be the event "the maximum of the two balls is at least equal to k". Compute $\mathbb{P}(C_k)$ and justify your answer.

4) Let E be the event "the first ball is not 1", let F be the event "the second ball is not n". Compute $\mathbb{P}(E|F)$. Are the events E and F independent? Justify your answers.

Part 4

Exercise 6. Let $n \geq 1$ be an integer and denote by $\mathcal{S}_n$ the set of all permutations of $\{1, 2, 3, \ldots, n\}$. We say that $\sigma \in \mathcal{S}_n$ is an involution if $\sigma \circ \sigma = \mathrm{Id}$, where Id is the identity permutation. Denote by I_n the number of involutions of $\mathcal{S}_n$. We set $I_0 = 1$ by convention.

1) For every permutation of $\mathcal{S}_3$, check if it is an involution and deduce the value of I_3.

2) Is the product of two involutions of $\mathcal{S}_n$ always an involution of $\mathcal{S}_n$? Justify your answer.

3) Show that for every $n \geq 1$, $I_{n+1} = I_n + nI_{n-1}$.

4)

a) Show (carefully) that any involution of $\mathcal{S}_n$ can be written as a product of transpositions with disjoint supports (by convention, we say that the identity permutation is a product of 0 transpositions with disjoint supports).

b) Show that

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot \frac{(2k)!}{k! \cdot 2^k},$$

where $\lfloor n/2 \rfloor$ is the greatest integer at most equal to $n/2$ and $0! = 1$.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let $n \geq 2$ be an integer and denote by $\mathcal{S}_n$ the set of all permutations of $\{1, 2, 3, \ldots, n\}$. Let $\Phi : \mathcal{S}_n \to \mathbb{C}\backslash\{0\}$ be a function such that $\Phi(\sigma \circ \tau) = \Phi(\sigma)\Phi(\tau)$ for every $\sigma, \tau \in \mathcal{S}_n$. Show that either Φ is equal to the identity, or to the signature.

Exercise 8. What does the following image represent?