Week 17 (Final exam): Friday, February 1st, 14:00-16:00 pm

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Remark. *If at some point you do not manage to solve a question, you can write on your sheet of paper that you assume that it is true and use it to solve a next question if needed.*

Part 1

Exercise 1.

1) Give the definition of a probability on a finite state space Ω.

2) Let $(\Omega, \mathbb{P})$ be a finite probability space. Let B be an event such that $\mathbb{P}(B) > 0$. Show that the function

$$\begin{array}{cccc} \mathbb{P}_B & : & \mathcal{P}(\Omega) & \rightarrow & [0,1] \\ & & A & \mapsto & \mathbb{P}(A|B) \end{array}$$

is a probability on Ω.

Solution of exercise 1.

1) A probability on a finite state space Ω is a function $\mathbb{P} : \mathcal{P}(\Omega) \rightarrow [0,1]$ such that $\mathbb{P}(\Omega) = 1$ and such that for every $A, B \subset \Omega$ such that $A \cap B = \varnothing$ we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

2) We first compute $\mathbb{P}_B(\Omega) = \mathbb{P}(\Omega|B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$. Next, take $A_1, A_2 \subset \Omega$ such that $A_1 \cap A_2 = \varnothing$. We compute $\mathbb{P}_B(A_1 \cup A_2)$:

$$\mathbb{P}_B(A_1 \cup A_2) = \frac{\mathbb{P}((A_1 \cup A_2) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}((A_1 \cap B) \cup (A_2 \cap B))}{\mathbb{P}(B)}.$$

But since $A_1 \cap B$ and $A_2 \cap B$ are disjoint and since $\mathbb{P}$ is a probability, we get

$$\mathbb{P}_B(A_1 \cup A_2) = \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} + \frac{\mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} = \mathbb{P}_B(A_1) + \mathbb{P}_B(A_2).$$

□

Exercise 2. Consider the following three assertions:

(A) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$, (B) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0$, (C) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0$.

For each one of the assertions, say if it is true or false (and justify your answer).

Solution of exercise 2.

(A) is true. Indeed, fix $x \in \mathbb{R}$ and take $y = -x+1$. Then $x+y > 0$.

(B) is false. Its negation, $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y \leq 0$ is true. Indeed, fix $x \in \mathbb{R}$ and take $y = -x-1$. Then $x+y \leq 0$.

(C) is false. Its negation, $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y \leq 0$ is true. Indeed, we may take $x = -1$ and $y = -1$, so that $x+y \leq 0$. □

Part 2

Exercise 3. Let $n \geq 1$ be an integer.

1) Show that $\binom{2n+2}{n+1} = \left(4 - \frac{2}{n+1}\right)\binom{2n}{n}$.

2) Show that $\binom{2n}{n} < 4^n$.

Solution of exercise 3.

1) Write

$$\binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!^2} = \frac{(2n+2)(2n+1)}{(n+1)^2}\frac{(2n)!}{n!^2} = \frac{2(2n+1)}{n+1}\binom{2n}{n}.$$

Since $4 - \frac{2}{n+1} = \frac{4n+2}{n+1} = \frac{2(2n+1)}{n+1}$, this completes the proof.

2) We argue by induction. Set $P(n)$: "$\binom{2n}{n} < 4^n$".

Basis step. For $n = 1$, $\binom{2n}{n} = 2 < 4$.

Inductive step. Fix $n \geq 1$ such that $P(n)$ is true. We show that $P(n+1)$ is true. Using the fact that $4 - \frac{2}{n+1} < 4$ and the fact that $P(n)$ is true, we get $\binom{2n+2}{n+1} < 4 \cdot 4^n = 4^{n+1}$.

We conclude that $P(n)$ is true for every $n \geq 1$. □

Exercise 4. Let E be a set and let $f : E \to E$ be a function such that $f \circ f \circ f = f$.

1) Show that f is onto if and only if f is one-to-one.

2) Give an example of a set E and of a function $f : E \to E$ be a function such that $f \circ f \circ f = f$ which is neither onto, neither one-to-one (and explain why it is neither onto, neither one-to-one).

Solution of exercise 4.

1) We argue by double implication.

First assume that f is one-to-one. Fix $x \in E$. Then $f(f(f(x)) = f(x)$. Hence $f(f(x)) = x$. Hence $f(x)$ is a preimage of x. Hence f is onto.

Now assume that f is onto. We show that $f(f(y)) = y$ for every $y \in E$. To this end, fix $y \in E$. Since f is onto, there exists $x \in E$ such that $y = f(x)$. Writing $f(f(f(x))) = f(x)$ gives $f(f(y)) = y$. The fact that f is one-to-one readily follows: fix $x, y \in E$ such that $f(x) = f(y)$. Then $x = f(f(x)) = f(f(y)) = y$, so that $x = y$.

Alternative solution for the second implication: assume that f is onto. To show that f is onto, fix $x,y \in E$ such that $f(x) = f(y)$. Since f is onto, we may find $u,v \in E$ such that $x = f(u)$ and $y = f(v)$. Then $f(f(u)) = f(f(v))$. By composing by f we get $x = f(u) = f(f(f(u))) = f(f(f(v))) = y$. Hence $x = y$.

2) Take $E = \{1,2\}$ and $f : E \to E$ to be the function defined by $f(1) = f(2) = 1$. Then $f(f(f(x))) = f(x) = 1$ for every $x \in E$ and is neither onto (2 has no preimage), nor one-to-one (since $f(1) = f(2)$). $\square$

Part 3

Exercise 5. Let $n \geq 2$ be an integer. We consider an urn with n numbered balls from 1 to n. We choose at random one ball, put it back in the urn, and then choose at random another ball. We model this experiment with the state space $\Omega = \{1,2,\ldots,n\}^2$ equipped with the uniform probability $\mathbb{P}$. Fix an integer $k \in \{1,2,\ldots,n\}$.

1) Let A_k be the event "the maximum of the two balls is less than or equal to k". Compute $\mathbb{P}(A_k)$ and justify your answer.

2) Let B_k be the event "the maximum of the two balls is equal to k". Compute $\mathbb{P}(B_k)$ and justify your answer.

3) Let C_k be the event "the maximum of the two balls is at least equal to k". Compute $\mathbb{P}(C_k)$ and justify your answer.

4) Let E be the event "the first ball is not 1", let F be the event "the second ball is not n". Compute $\mathbb{P}(E|F)$. Are the events E and F independent? Justify your answers.

Solution of exercise 5.

1) Note that A_k is the event "both balls are less than or equal to k", so that $A_k = \{1,2,\ldots,k\}^2$. Therefore $\mathbb{P}(A_k) = \frac{k^2}{n^2}$.

2) We have $B_k = \{(1,k),(2,k),\ldots,(k-1,k),(k,1),\ldots,(k,k-1),(k,k)\}$, so that $\mathrm{Card}(B_k) = 2(k-1)+1 = 2k-1$. Therefore $\mathbb{P}(B_k) = \frac{2k-1}{n^2}$.

3) Note that the complement of the event C_k is A_{k-1}. Therefore $\mathbb{P}(C_k) = 1 - \mathbb{P}(A_{k-1}) = 1 - \frac{(k-1)^2}{n^2}$ (this formula is also valid for $k = 1$).

4) Note that $\mathrm{Card}(F) = n(n-1)$ (n choices for the first ball, $n-1$ choices for the second ball) and that $\mathrm{Card}(E \cap F) = (n-1)^2$ ($n-1$ choices for the first ball, $n-1$ choices for the second ball). Therefore

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{(n-1)^2/n^2}{n(n-1)/n^2} = \frac{n-1}{n}.$$

We have $\mathbb{P}(E) = \frac{n-1}{n} = \mathbb{P}(E|F)$ (or, equivalently, $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$), so the events E and F are independent. $\square$

Part 4

Exercise 6. Let $n \geq 1$ be an integer and denote by $\mathcal{S}_n$ the set of all permutations of $\{1, 2, 3, \ldots, n\}$. We say that $\sigma \in \mathcal{S}_n$ is an involution if $\sigma \circ \sigma = \mathrm{Id}$, where Id is the identity permutation. Denote by I_n the number of involutions of $\mathcal{S}_n$. We set $I_0 = 1$ by convention.

1) For every permutation of $\mathcal{S}_3$, check if it is an involution and deduce the value of I_3.

2) Is the product of two involutions of $\mathcal{S}_n$ always an involution of $\mathcal{S}_n$? Justify your answer.

3) Show that for every $n \geq 1$, $I_{n+1} = I_n + nI_{n-1}$.

4)

a) Show (carefully) that any involution of $\mathcal{S}_n$ can be written as a product of transpositions with disjoint supports (by convention, we say that the identity permutation is a product of 0 transpositions with disjoint supports).

b) Show that

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot \frac{(2k)!}{k! \cdot 2^k},$$

where $\lfloor n/2 \rfloor$ is the greatest integer at most equal to $n/2$ and $0! = 1$.

Solution of exercise 6.

1) Consider the 6 permutations of $\mathcal{S}_3$ $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $\sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. We check that $\sigma_1, \sigma_2, \sigma_3, \sigma_6$ are involutions. However $\sigma_4^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\sigma_5^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ are not involutions. Hence $I_3 = 4$.

2) Yes for $n = 1$ and for $n = 2$ (all the permutations of $\mathcal{S}_1$ and of $\mathcal{S}_2$ are involutions). No for $n \geq 3$, since the product of the two transpositions $(1,2)$ and $(2,3)$ is $(2,3)(1,2) = (1,3,2)$ is not an involution (since $(1,3,2)^2 = (1,2,3)$).

3) The idea is to do a disjunction of cases according to the value of $\sigma(n+1)$. The number of involutions σ of $\mathcal{S}_{n+1}$ such that $\sigma(n+1) = n+1$ is I_n since choosing a permutation σ of $\mathcal{S}_{n+1}$ such that $\sigma(n+1) = n+1$ amounts to choosing an involution of $\mathcal{S}_n$. Also, the number of involutions σ of $\mathcal{S}_{n+1}$ such that $\sigma(n+1) = k$ (for a fixed $1 \leq k \leq n+1$) is I_{n-1}: indeed, to constructing an involution of σ of $\mathcal{S}_{n+1}$ such that $\sigma(n+1) = k$ amounts to choosing an involution of $\{1, 2, \ldots, k-1, k, \ldots, n\}$ (since $\sigma(k) = \sigma(\sigma(n+1)) = n+1$). By the sum rule, we get $I_{n+1} = I_n + nI_{n-1}$.

4) a)

First solution. Let σ be an involution of $\mathcal{S}_n$. We know that σ can be written (in a unique way) as a product of cycles with disjoint support. We argue by contradiction and assume that there exists at least one of these cycles $\mathcal{C}$ which has at least 3 elements in its support. Write $\mathcal{C} = (x_1, x_2, \ldots, x_p)$ with $p \geq 3$ and $x_1, \ldots, x_p$ different integers. Then $\sigma^2(x_1) = x_3 \neq x_1$, so σ is not an involution. This is a contradiction, and shows that all the cycles have 2 elements in their support, meanining that they are transpositions.

Second solution. We argue by strong induction on n.

Basis step. For $n=2$, the identity and the transposition $(1,2)$ are indeed products of transpositions with disjoint supports.

Inductive step. Let $n \geq 2$ be an integer and assume that all the involutions on sets of size $2,3,\dots,n$ can be written as products of transpositions with disjoint supports. Consider an involution $\sigma \in \mathcal{S}_{n+1}$.

First case: $\sigma(n+1)=n+1$. Then σ, restricted to $\{1,2,\dots,n\}$, is an involution on $\mathcal{S}_n$ that can be written as a product of transpositions with disjoint supports. This product also expresses $\sigma \in \mathcal{S}_{n+1}$ as a product of transpositions with disjoint supports.

Second case: $\sigma(n+1) \neq n+1$. Set $k=\sigma(n+1)$ and consider the permutation $\tau = \sigma \circ (k, n+1)$. Then $\tau(k)=k$ and $\tau(n+1)=n+1$, so we may consider the restriction of τ on the set $\{1,2,\dots,k-1,k+1,\dots,n\}$ of size $n-1$. By the induction hypothesis, we may write

$$\tau = (a_1,b_1)\cdots(a_r,b_r)$$

as a product of transpositions with disjoint supports and with $a_1,b_1,\dots,a_r,b_r$ different from k and $n+1$. it follows that

$$\sigma \circ (k,n+1) = (a_1,b_1)\cdots(a_r,b_r),$$

so that

$$\sigma = (a_1,b_1)\cdots(a_r,b_r)(k,n+1)$$

is a product of transpositions with disjoint supports.

b) The previous question invites us to distinguish according to the number of transpositions with disjoint support needed to factorize an involution (this number is well defined since we know that the factorization in cycles with disjoint supports is unique up to order of the factors).

More precisely, for fixed $0 \leq k \leq \lfloor n/2 \rfloor$, let us count the number of involutions which are the product of k transpositions with disjoint support. First, we choose the $2k$ integers which appear in the supports of the k transpositions ($\binom{n}{2k}$ choices). Next, we pair together these $2k$ integers. We claim that there are $\frac{(2k)!}{2^k k!}$ such pairings. Indeed, to construct such a pairing, we write these $2k$ integers in any order ($(2k)!$ possibilities) and pair together the first and second integers, then the third and fourth and so on. In this construction, every pairing is counted $2^k k!$ times (indeed, the integers in each pair can be exchanged, which gives the factor 2^k, and the k pairs can be permuted, which gives the factor $k!$). We conclude that

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{n!}{k!(n-2k)!}.$$

□

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let $n \geq 2$ be an integer and denote by $\mathcal{S}_n$ the set of all permutations of $\{1, 2, 3, \dots, n\}$. Let $\Phi : \mathcal{S}_n \to \mathbb{C}\backslash\{0\}$ be a function such that $\Phi(\sigma \circ \tau) = \Phi(\sigma)\Phi(\tau)$ for every $\sigma, \tau \in \mathcal{S}_n$. Show that either Φ is equal to the identity, or to the signature.

Solution of exercise 7. First, by taking σ and τ to be the identity permutation Id, we get $\Phi(\mathrm{Id}) = \Phi(\mathrm{Id})^2$. Since $\Phi(\mathrm{Id}) \neq 0$, we conclude that $\Phi(\mathrm{Id}) = 1$.

Second, by taking σ and $\tau = \sigma^{-1}$, we get that $\Phi(\sigma)\Phi(\sigma^{-1}) = \Phi(\mathrm{Id}) = 1$.

Third, if τ is a permutation, we get $1 = \Phi(\mathrm{Id}) = \Phi(\tau \circ \tau) = \Phi(\tau)^2$. Therefore, either $\Phi(\tau) = 1$, either $\Phi(\tau) = -1$.

Fourth, we claim that all transpositions have the same image by Φ. Indeed, if $\pi = (a, b)$ and $\sigma = (c, d)$ are two transpositions, we consider a permutation τ such that $\tau(a) = c$ and $\tau(b) = d$. Then it is readily checked that $\tau \circ \pi \circ \tau^{-1} = \sigma$. It follows that

$$\Phi(\pi) = \Phi(\tau)\Phi(\pi)\Phi(\tau^{-1}) = \Phi(\tau \circ \pi \circ \tau^{-1}) = \Phi(\sigma).$$

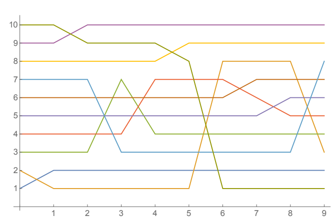
Finally:

– if $\Phi((1, 2)) = 1$, then $\Phi(\tau) = 1$ for every transposition τ. Since every permutation is a product of transpositions, we conclude that Φ is the identity.‘

– if $\Phi((1, 2)) = -1$, then $\Phi(\tau) = -1$ for every transposition τ. Therefore, if a permutation σ is a product of r transpositions, we conclude that $\Phi(\sigma) = (-1)^r$, which is precisely the value of the signature of σ.

This completes the proof. □

Exercise 8. What does the following image represent?



Solution of exercise 8. This image represents the cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ as a product of 9 transpositions: $(3, 8)(5, 6)(6, 7)(1, 8)(8, 9)(4, 7)(3, 7)(9, 10)(1, 2)$. □