Week 16 (Final exam): Monday, January 29th, 13:15-15:15 pm

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1. 1) Let $(\Omega, \mathbb{P})$ be a finite probability space. Show that for every $A, B \subset \Omega$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Remark. You may use without proof the fact that if $(A_i)_{1 \leq i \leq n}$ are pairwise disjoint events, then $\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$.

2) Give the definition of the signature of a permutation. What ways do you know to compute the signature of a permutation?

Solution of exercise 1. 1) Write $A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$, which is a union of pairwise disjoint events. Hence

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \backslash B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \backslash A).$$

But $(A \backslash B) \cup (A \cap B) = A$ is a union of disjoint events, hence $\mathbb{P}(A \backslash B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$. Similarly, $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Therefore

$$\mathbb{P}(A \cup B) = (\mathbb{P}(A) - \mathbb{P}(A \cap B)) + \mathbb{P}(A \cap B) + (\mathbb{P}(B) - \mathbb{P}(A \cap B)),$$

and the desired result follows.

2) If $\sigma \in S_n$, the signature of σ is defined by $\epsilon(\sigma) = (-1)^{I(\sigma)}$, where $I(\sigma)$ is the number of inversions of σ (we say that (i, j) is an inversion if $i < j$ and $\sigma(i) > \sigma(j)$).

To compute the signature of the permutation, we can either find its number of inversions, or try to factorize it into "simpler" permutations and use one of the following results:

a) For every $\pi, \sigma \in S_n$, $\epsilon(\pi\sigma) = \epsilon(\pi)\epsilon(\sigma)$.

b) If $\sigma = \tau_1 \cdots \tau_N$ is a product of transpositions, then $\epsilon(\sigma) = (-1)^N$.

c) Is σ is a p-cycle, then $\epsilon(\sigma) = (-1)^{p+1}$. □

Exercise 2. 1) Give an example of sets E, F and an assertion $P(x, y)$ (with $x \in E$ and $y \in F$) such that the assertion "$\forall x \in E,\ \exists y \in F,\ P(x, y)$" is true but the assertion "$\exists x \in E,\ \forall y \in F,\ P(x, y)$" is false. Justify your answer.

2) Can one find sets E, F and an assertion $P(x, y)$ (with $x \in E$ and $y \in F$) such that the assertion "$\exists x \in E,\ \forall y \in F, P(x, y)$" is true but the assertion "$\forall x \in E,\ \exists y \in F, P(x, y)$" is false? Justify your answer.

Solution of exercise 2.

1) Take $E = F = \mathbb{R}$ and $P(x,y)$: "$x = y$". Then $\forall x \in \mathbb{R}, \exists y \in R, x = y$ is true (once is x fixed, we may take $y = x$), but $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x = y$ is false (indeed, if such an x exists, take $y = x + 1$, and $x = y$ is not true).

2) Yes. Take $E = F = \mathbb{R}$ and $P(x,y)$: "$x(y^2 + 1) = 0$". Then $\exists x \in E, \forall y \in F, x(y^2 + 1) = 0$ is true (take $x = 0$). But $\forall x \in E, \exists y \in F, x(y^2+1) = 0$ is false (if $x = 1$, one cannot find $y \in \mathbb{R}$ such that $y^2+1 = 0$). □

Part 2

Exercise 3. Let A and B be two subsets of a set E.

1) Show (carefully) that $A \subset B \iff A \cup B = B$.

2) Show (carefully) that $A = B \iff A \cap B = A \cup B$.

Solution of exercise 3.

1) We show the double implication.

▹ Assume that $A \subset B$. We show that $A \cup B = B$ by double inclusion. First, if $x \in A \cup B$, then either $x \in B$, or $x \in A$, in which case we also have $x \in B$ (because $A \subset B$). Second, we clearly have $B \subset A \cup B$. Therefore $A \cup B = B$.

▹ Assume that $A \cup B = B$. We show that $A \subset B$. To this end, take $x \in A$. Then $x \in A \cup B = B$, so that $x \in B$.

2) We show the double implication.

▹ Assume that $A = B$. Then $A \cap B = A = A \cup B$.

▹ Assume that $A \cap B = A \cup B$. We show that $A = B$ by double inclusion. First, if $x \in A$, then $x \in A \cap B$, hence $x \in A \cup B$ and thus $x \in B$. Therefore $A \subset B$. One shows that $B \subset A$ in the same way. □

Exercise 4. 1) Let $\mathcal{S} = \{(u_n)_{n \geq 0}\}$ be the set of all sequences of real numbers. Let $F : \mathcal{S} \to \mathcal{S}$ be the function defined by $F : (u_0, u_1, u_2, u_3 \dots) \mapsto (u_0, u_2, u_4, u_6, u_8, \dots)$. Is F one-to-one? Onto? Justify your answers.

2) Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $G : (a, b, c) \mapsto (2b, -c, a/2)$. Show that G is a bijection and give the expression of G^{-1}. Justify your answer.

Solution of exercise 4.

1)

▹ F is not one-to-one, since

$$F(1,2,1,2,1,2,1,2,\dots) = F(1,1,1,1,1,1,1,1,\dots) = (1,1,1,1,1,1,1,1,1,\dots).$$

▹ F is onto. Indeed, let $\mathbf{u} = (u_0, u_1, u_2, \dots)$ be any sequence in $\mathcal{S}$. Then

$$F(u_0, 0, u_1, 0, u_2, 0, u_3, 0, u_4, \dots) = \mathbf{u}.$$

The function F is not a bijection.

2)

▹ G is one-to-one. Indeed, fix $(a,b,c) \in \mathbb{R}^3$ and $(a',b',c') \in \mathbb{R}^3$ such that $G(a,b,c) = G(a',b',c')$. Hence $2b = 2b'$, $-c = -c'$ and $a/2 = a'/2$, so that $(a,b,c) = (a',b',c')$.

▹ G is onto. Indeed, if $(a,b,c) \in \mathbb{R}^3$, $G(2c, a/2, -b) = (a,b,c)$.

We conclude that G is a bijection, and that $G^{-1}(a,b,c) = (2c, a/2, -b)$ for every $a,b,c \in \mathbb{R}^3$.

□

Part 3

Exercise 5. Fix $p \in [0,1]$. Sophia flips a coin three times in a row. We assume that the coin is unfair: it gives 'Heads' with probability p, and 'Tails' with probability $1-p$. We assume that we are given a finite probability space $(\Omega, \mathbb{P})$ with, for $1 \leq i \leq 3$, events $H_i = \{\text{the } i\text{-th result is 'Heads'}\}$ such that $\mathbb{P}(H_i) = p$ for $1 \leq i \leq 3$ and such that H_1, H_2, H_3 are independent.

1) For each one of the following events, write the event using only symbols $H_1, H_2, H_3, \overline{H_1}, \overline{H_2}, \overline{H_3}, \cap, \cup$:

a) $E = \{\text{None of the result is 'Heads'}\}$ b) $F = \{\text{(At least) the two first results are identical}\}$

c) $G = \{\text{We obtain (at least) two consecutive 'Heads'}\}$.

2) Compute $\mathbb{P}(E)$, $\mathbb{P}(F)$, $\mathbb{P}(G)$.

Solution of exercise 5.

1)

a) $E = \overline{H_1} \cap \overline{H_2} \cap \overline{H_3}$.

b) $F = (H_1 \cap H_2) \cup (\overline{H_1} \cap \overline{H_2})$

c) G=$(H_1 \cap H_2) \cup (H_2 \cap H_3)$

2)

a) By independence, $\mathbb{P}(E) = \mathbb{P}(\overline{H_1})\mathbb{P}(\overline{H_2})\mathbb{P}(\overline{H_3}) = (1-p)^3$.

b) The union defining F is disjoint, hence

$$\mathbb{P}(F) = \mathbb{P}(H_1 \cap H_2) + \mathbb{P}(\overline{H_1} \cap \overline{H_2}) = p^2 + (1-p)^2,$$

where the last equality has been obtained by using independence.

c) The difficulty is the union defining G is not disjoint. Write

$$\begin{aligned}
\mathbb{P}(G) &= \mathbb{P}(H_1 \cap H_2) + \mathbb{P}(H_2 \cap H_3) - \mathbb{P}\big((H_1 \cap H_2) \cap (H_2 \cap H_3)\big) \\
&= \mathbb{P}(H_1 \cap H_2) + \mathbb{P}(H_2 \cap H_3) - \mathbb{P}(H_1 \cap H_2 \cap H_3) \\
&= 2p^2 - p^3,
\end{aligned}$$

where the last equality follows from independence. □

Part 4

Exercise 6. Let $n \geq 1$ be an integer.

1) (example) For $j = 2$ and $n = 3$, check that $\sum_{k=j}^{n} \binom{k-1}{j-1} = \binom{n}{j}$

2) Let P_n be the property "For every integer $1 \leq j \leq n$, $\sum_{k=j}^{n} \binom{k-1}{j-1} = \binom{n}{j}$". Show that P_n is true by induction.

We denote by S_n the set of all permutations of $\{1, 2, \dots, n\}$ and we say that a permutation $\sigma \in S_n$ has a record at $j \in \{1, \dots, n\}$ if $\sigma(i) < \sigma(j)$ for every positive integer i such that $i < j$.

3) Fix two positive integers j, k such that $1 \leq j \leq k \leq n$. Justify that there are $\binom{k-1}{j-1}(j-1)!(n-j)!$ permutations $\sigma \in S_n$ such that both $\sigma(j) = k$ and σ has a record at j.

4) Let $\mathbb{P}$ be the uniform probability on S_n. For $1 \leq j \leq n$, give a simple expression of the probability of the event $\left\{\sigma \in S_n : \sigma \text{ has a record at } j\right\}$.

Hint. If you do not manage to solve one of the previous questions, you can assume that the results of the previous questions are true in order to solve this question.

Solution of exercise 6.

1) We have $\sum_{k=2}^{3} \binom{k-1}{1} = \binom{1}{1} + \binom{2}{1} = 1 + 2 = 3$ and $\binom{3}{2} = 3$.

2) We argue by induction.

▷ *Basis step.* For $n = 1$, and $j = 1$, we have $\sum_{k=1}^{1} \binom{k-1}{0} = 1$ and $\binom{1}{1} = 1$.

▷ *Inductive step.* Fix an integer $n \geq 1$ such that P_n is true. We show that P_{n+1} is true. To this end, fix $1 \leq j \leq n+1$.

First case: $j = n+1$. Then $\sum_{k=n+1}^{n+1} \binom{k-1}{j-1} = \binom{n}{n} = 1 = \binom{n+1}{n+1}$.

Second case: $0 \leq j \leq n$. Then, since P_n is true,

$$\sum_{k=j}^{n+1} \binom{k-1}{j-1} = \left(\sum_{k=j}^{n} \binom{k-1}{j-1} \right) + \binom{n}{j-1} = \binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j},$$

where the last identity follows from Newton's relation.

3) Choosing a permutation $\sigma \in S_n$ such that both $\sigma(j) = k$ and σ has a record at j amounts to choosing:

- the set A of images of the first $j-1$ integers in $\{1, 2, \dots, k-1\}$ (so that j is a record), hence $\binom{k-1}{j-1}$ choices,
- a bijection from $\{1, 2, \dots, j-1\}$ to A, hence $(j-1)!$ choices,
- a bijection from $\{j+1, \dots, n\}$ to $\{1, 2, \dots, n\} \setminus (A \cup \{k\})$, hence $(n-j)!$ choices.

We get the desired result by the "multiplicative principle".

4) The set $\{\sigma \in S_n : \sigma \text{ has a record at } j\}$ can be written as the disjoint union

$$\{\sigma \in S_n : \sigma \text{ has a record at } j\} = \bigcup_{k=1}^{n} \{\sigma \in S_n : \sigma \text{ has a record at } j \text{ and } \sigma(j) = k\}.$$

Therefore

$$\frac{\mathrm{Card}(\{\sigma \in S_n : \sigma \text{ has a record at } j \text{ and } \sigma(j) = k\})}{n!} = \frac{\binom{k-1}{j-1}(j-1)!(n-j)!}{n!}$$

thank to question 3). Since $\mathbb{P}$ is the uniform probability on S_n, using question 2), we conclude that

$$\begin{aligned}
\mathbb{P}\left(\left\{\sigma \in S_n : \sigma \text{ has a record at } j\right\}\right) &= \frac{\mathrm{Card}\left(\left\{\sigma \in S_n : \sigma \text{ has a record at } j\right\}\right)}{\mathrm{Card}(S_n)} \\
&= \frac{1}{n!} \sum_{k=j}^{n} \binom{k-1}{j-1} (j-1)!(n-j)! \\
&= \frac{(n-j)!(j-1)!}{n!} \sum_{k=j}^{n} \binom{k-1}{j-1} \\
&= \frac{(n-j)!(j-1)!}{n!} \binom{n}{j} = \frac{1}{j}.
\end{aligned}$$

□

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let $n \geq 2$ be an integer. We say that a permutation σ of $\{1, 2, \ldots, n\}$ is decomposable if there exists an integer $1 \leq k \leq n-1$ such that $\sigma(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$. Denote by d_n the number of decomposable permutations σ of $\{1, 2, \ldots, n\}$ and let $p_n = \frac{d_n}{n!}$ the probability that permutation of S_n chosen uniformly at random is decomposable. Find an asymptotic equivalent of p_n when $n \to \infty$ (that is, the simplest expression as possible a_n such that $p_n / a_n \to 1$ when $n \to \infty$).

Solution of exercise 7. Every decomposable permutation σ of S_n is obtained from a non-decomposable permutation of $\{1, 2, \ldots, k\}$ for a certain integer $1 \leq k \leq n-1$ followed by any permutation of $\{k+1, k+2, \ldots, n\}$. Writing c_n for the number of non-decomposable permutations of length n, the previous remark allows to write

$$n! - c_n = \sum_{k=1}^{n-1} c_k (n-k)!.$$

Hence

$$p_n = \frac{n! - c_n}{n!} = \sum_{k=1}^{n-1} \frac{c_k}{k!} \frac{1}{\binom{n}{k}} = \sum_{k=1}^{n-1} (1 - p_k) \frac{1}{\binom{n}{k}}.$$

▷ We first show that $p_n \to 0$. Since $p_1 = 0$, and, for, $2 \le k \le n-2$, $\binom{n}{k} \ge \frac{n(n-1)}{2}$, we get that for $n \ge 4$,

$$\begin{aligned} p_n &= \frac{1}{n} + (1 - p_{n-1}) \frac{1}{n} + \sum_{k=2}^{n-2} (1 - p_k) \frac{1}{\binom{n}{k}} \\ &\le \frac{1}{n} + \frac{1}{n} + n \frac{2}{n(n-1)}. \end{aligned}$$

Since $p_n \ge 0$, this establishes that $p_n \to 0$

▷ For $n \ge 6$,

$$p_n = \frac{1}{n} + (1 - p_2) \frac{2}{n(n-1)} + (1 - p_{n-1}) \frac{1}{n} + (1 - p_{n-2}) \frac{2}{n(n-1)} + \sum_{k=3}^{n-3} (1 - p_k) \frac{1}{\binom{n}{k}}$$

But $\binom{n}{k} \ge \frac{n(n-1)(n-2)}{6}$ for $3 \le k \le n-3$. Hence

$$2 - p_{n-1} + \frac{2(2 - p_2 - p_{n-2})}{n-1} \le n p_n \le 2 - p_{n-1} + \frac{2(2 - p_2 - p_{n-2})}{n-1} + n^2 \frac{6}{n(n-1)(n-2)}.$$

Since $p_n \to 0$, we get that $np_n \to 2$, or, in other words, $p_n \sim \frac{2}{n}$.

□

Exercise 8. Let $n \ge 1$ be an integer. An urn contains n blue balls and n red balls. Choose successively without replacement balls at random until there is only one colour left in the urn. Denote by H_n the number of balls that then remain in the urn. For every $0 \le k \le n$, compute $\mathbb{P}(H_n = k)$.

You can express your result using binomial coefficients.

Solution of exercise 8. Fix $k \in \{1, 2, \ldots, n\}$. Then k red balls are left in the urn if and only if the $(2n - k)^{\text{ième}}$ draw gives a red ball, and all the other $n-1$ red balls have already been chosen during the first $2n - k - 1$ draws. Thus there are $\binom{2n-k-1}{n-1}$ such configurations. By symmetry, the probability that k blue balls are left is the same as the probability that there are k red balls. Hence,

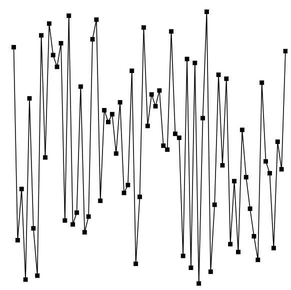
$$\mathbb{P}(H_n = k) = 2 \frac{\binom{2n-k-1}{n-1}}{\binom{2n}{n}}.$$

Remarque. By Stirling's formula,

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \qquad \binom{2n-k-1}{n-1} \sim \frac{1}{2^{k+1}} \frac{4^n}{\sqrt{\pi n}}.$$

so that for fixed $k \geq 1$, as $n \to \infty$, we have $\mathbb{P}(H_n = k) \to \frac{1}{2^k}$. □

Exercise 9. What does the following image represent, and how was it obtained?



Solution of exercise 9. This image depicts a random permutation. More precisely, for $n = 70$, we have chosen $\sigma \in S_{70}$ uniformly at random, drawn squares centered at all points with coordinates $(i, \sigma(i))$ for $1 \leq i \leq 70$, and straight lines between the points $(i, \sigma(i))$ and $(i+1, \sigma(i+1))$ for $1 \leq i \leq 69$. See the code below that generated this image (using the software Mathematica). □

```
n = 70;
permutation = RandomSample[Range[n]];
squares = Rectangle[{#, permutation[[#]]}] & /@ Range[n];
lines =
  Line[{{# + 1 / 2, permutation[[#]] + 1 / 2}, {# + 3 / 2, permutation[[# + 1]] + 1 / 2}}] & /@
   Range[n - 1];
Graphics[{squares, lines}]
```

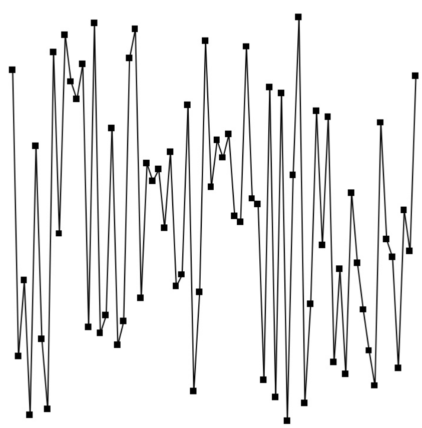


Figure 1: The Mathematica code used to generate the image of exercise 9.