CORRIGENDUM TO "CRITICAL TREES ARE NEITHER TOO SHORT NOR TOO FAT"

ABSTRACT. A formal verification in the Lean proof assistant of the results of *Critical trees are neither too short nor too fat* (arXiv:2311.06163v2, https://arxiv.org/abs/2311.06163) conducted by Igor Kortchemski found the errors listed below. None of them affects the main results (Theorems 1.1, 1.2, 1.3 and Corollary 1.4), which have been formally verified, nor any other numbered result except as corrected below. The formalization is available at https://github.com/igor-kortchemski/critical-trees-height-width.

Locations refer to the numbering and pagination of arXiv:2311.06163v2. Each item gives the location, the correction, and the reason. The corrections are grouped into statements and definitions, repairs to proofs, and local mathematical corrections. Within each section, items follow the order of their first occurrence in the paper.

1. CORRECTIONS TO STATEMENTS AND DEFINITIONS

(1) *Third example: Introduction, p. 3, and Section 3.6, p. 14.*
With $L(n) \sim \frac{1-\beta}{\beta} \ln(n)^{-\beta} e^{-\ln(n)^\beta}$ as printed,

$$\ell_\star(s) \sim \frac{1-\beta}{\beta^2} \ln(s)^{1-2\beta} e^{-\ln(s)^\beta}, \qquad V(x) \sim \frac{\beta}{1-\beta} \ln(x)^\beta e^{\ln(x)^\beta},$$

not $\ell_\star(s) \sim \frac{1}{\beta} \ln(s)^{1-\beta} e^{-\ln(s)^\beta}$ and $V(x) \sim e^{\ln(x)^\beta}$; in particular the stated equivalent of h_n is off by an unbounded factor. The smallest correction is to replace the hypothesis on L, in both places, by

$$L(n) \sim e^{-\ln(n)^\beta},$$

for which the printed $\ell_\star$, V, b_n, the expansion of h_n and the equivalent $b_n V(b_n) \sim \frac{n}{\beta} \ln(n)^{1-\beta}$ (p. 15) hold as stated. *Reason.* By (3.6), $\ell_\star(e^u) \sim \int_u^\infty L(e^v)\, dv$; with the printed L the substitution $w = v^\beta$ gives $\frac{1-\beta}{\beta^2} \int_{u^\beta}^\infty w^{1/\beta - 2} e^{-w}\, dw \sim \frac{1-\beta}{\beta^2} u^{1-2\beta} e^{-u^\beta}$, and $V(e^u) = \int_0^u dv/\ell_\star(e^v)$ follows. Conversely $-\frac{d}{du}\big(\frac{1}{\beta} u^{1-\beta} e^{-u^\beta}\big) = e^{-u^\beta}\big(1 - \frac{1-\beta}{\beta} u^{-\beta}\big) \sim e^{-u^\beta}$, so the printed $\ell_\star$ corresponds to $L(n) \sim e^{-\ln(n)^\beta}$.

(2) *Equation (2.1), p. 5.* The upper bound misses the queue offset: it should read

$$\Delta(t) \le \mathsf{Width}(t) \le 1 + \max\big(\mathsf{W}_i^{\mathsf{bfs}}(t),\ 0 \le i < |t|\big).$$

Reason. For the one-vertex tree, $\mathsf{Width} = 1$ while $\max \mathsf{W}^{\mathsf{bfs}} = 0$; for a star with k leaves, $\mathsf{Width} = k$ while $\max \mathsf{W}^{\mathsf{bfs}} = k - 1$. The additional 1 is harmless in Corollaries 3.3 and 4.4, which are asymptotic.

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(3) *Section 2.2, definition of the Vervaat transform, p. 6.* The increments must be read starting just after the time m of the first minimum: replace $x_{m+i \bmod [n]}$ by $x_{m+1+i \bmod [n]}$, that is,

$$\mathcal{V}(\mathbf{s})_{i+1} - \mathcal{V}(\mathbf{s})_i = x_{(m+i \bmod n)+1}, \qquad 0 \le i < n.$$

Reason. For the bridge $\mathbf{s} = (0, -1, 0, -1)$ we have $m = 1$ and $(x_1, x_2, x_3) = (-1, 1, -1)$; the printed formula gives first step $x_1 = -1$, so $\mathcal{V}(\mathbf{s})$ is not an excursion, whereas the corrected one gives $(0, 1, 0, -1)$.

(4) *Equation (3.2), p. 7.* With $b_n = -n\mathbb{E}[X\mathbf{1}_{|X|\le a_n}]$ as in (3.1) and $\mathcal{C}_1$ defined by $\mathbb{E}[e^{-\lambda \mathcal{C}_1}] = e^{\lambda \ln \lambda}$, the limit in (3.2) is $\mathcal{C}_1 + 1 - \gamma$, where γ is Euler's constant, not $\mathcal{C}_1$. Either replace $\mathcal{C}_1$ by $\mathcal{C}_1 + 1 - \gamma$ in (3.2), or replace b_n by $b_n - (1 - \gamma)a_n$ there. *Reason.* By the Lévy–Khintchine formula with Lévy measure $x^{-2}\,dx$ on $(0, \infty)$ and truncation at 1 (the scale of a_n), the limit Y satisfies $\log \mathbb{E}[e^{-\lambda Y}] = \lambda \ln \lambda + c\lambda$ with $c = \int_0^\infty (e^{-y} - 1 + y\mathbf{1}_{y\le 1})y^{-2}\,dy = \gamma - 1$. Only consequences that are insensitive to a shift of order $a_n = o(b_n)$ are used later, so nothing else changes.

(5) *Lemma 5.3, p. 23.* The quotient in the statement requires a positive denominator. The inequality can also be stated with the denominator multiplied into the left-hand side, which covers the case of a zero denominator. *Reason.* For $\mu_0 = \mu_1 = 1/2$, $N = 1$ and $n = 3$, the conditioned tree exists, but $\mathbb{P}(X_1 = N) = 0$. The cross-multiplied inequality in this case reads $0 \le \mathbb{P}(X_1^{\le N} + \cdots + X_n^{\le N} = -1)$.

(6) *Lemma 5.5 and its proof, pp. 26–27.* The normalization in the conclusion is wrong; the statement should read

$$\sup_{k\in\mathbb{Z}} \left| \sigma\sqrt{n}\,\mathbb{P}(S_n = k) - \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{(k - n\mu)^2}{2\sigma^2 n}\Big) \right| \le \frac{C}{\sqrt{n}}.$$

Reason. For X uniform on $\{-1, 0\}$ (which satisfies the hypotheses for every $\gamma < 1/2$, $\alpha > 1$, $\beta > 1$), $\mu = -1/2$, $\sigma = 1/2$ and, for even n and $k = -n/2$, $\sqrt{n}\,\mathbb{P}(S_n = k) \to \sqrt{2/\pi}$, whereas $\phi(0) = 1$. In the proof, the Fourier inversion formulas carry the factors $1/(\sigma\sqrt{n})$ (Jacobian) and $1/\sqrt{2\pi}$, and the cutoff is reversed: Petrov's bound holds for $|t| \le \frac{\sqrt{n}}{4}\,\sigma^3/\mathbb{E}|X-\mu|^3$, a smaller range than $|t| \le \sqrt{n}/4$ since $\sigma^3 \le \mathbb{E}|X-\mu|^3$. The argument works with the cutoff $c_0\sqrt{n}$, where $c_0 = c_0(\alpha, \beta, \gamma) > 0$ is a uniform lower bound for $\sigma^3/(4\mathbb{E}|X - \mu|^3)$ (positive because $\sigma^2 \ge \gamma/2$ under condition 1 and $\mathbb{E}|X-\mu|^3$ is bounded under conditions 2–3). The Gaussian tail integral has integrand $e^{-t^2/2}$, and the residual sum in the decomposition of $g(t)$ runs over all $k \in \mathbb{Z}$. The proof of Lemma 5.4 only uses the consequence that $\mathbb{P}(S_n = k)$ is bounded above and below by constant multiples of $1/\sqrt{n}$ for $|k - n\mu| \le 1$, which still holds.

2. Repairs to proofs

(7) *Equation (3.4), p. 8.* The upper bound misses the edges joining v'_n and $[\![\varnothing, v'_n[\![$ to the roots of the grafted trees: it should read

$$\mathsf{H}(T'_n) \le \|v'_n\| + 1 + \max\Big(\max_{1\le i\le |S_{n-1}|} \mathsf{H}(\tau_i),\, \mathsf{H}(\mathcal{F}^n_{\mathrm{L}}),\, \mathsf{H}(\mathcal{F}^n_{\mathrm{R}})\Big).$$

Apply the forest identifications preceding (3.4) and the corrected bound on the event that $S_{n-1} \le 0$ and the appended increment corresponds to v'_n. This event has probability tending to 1 by the uniqueness and degree assertion at the start of Section 3.3. *Reason.* If T'_n is a star rooted at its centre, then v'_n is the root, $\|v'_n\| = 0$, every τ_i is a single vertex and the forests are empty, so the printed right-hand side is 0 while $\mathsf{H}(T'_n) = 1$. The additional 1 is negligible after division by $h_n \to \infty$. To see why the event restriction is needed, take $(S_0, \dots, S_4) = (0, -1, 0, 0, -1)$. After appending the terminal value -1, the Vervaat transform has increments $(1, 0, -1, 0, -1)$. The root is the unique maximum-degree vertex, τ_1 is a single vertex and the forests at the root are empty. The corrected right-hand side is 1 while $\mathsf{H}(T'_5) = 2$.

(8) *Proof of Theorem 1.3, last display, p. 11.* The bound of $\mathbb{P}(\max_{|S_{n-1}|<i\le|\underline{S}_{n-1}|} \mathsf{H}(\tau_i) \ge (1+\varepsilon)h_n)$ by $\mathbb{P}(\max_{1\le i\le b_n} \mathsf{H}(\tau_i) \ge (1+\varepsilon)h_n) + \mathbb{P}(|S_{n-1} - \underline{S}_{n-1}| > b_n)$ does not follow: the window $(|S_{n-1}|, |\underline{S}_{n-1}|]$ does not start at 1, and its position depends on the trees τ_i, so the maximum over it cannot be compared in law with the maximum over the first b_n trees. Replace the display by

$$\mathbb{P}\Big(\max_{|S_{n-1}|<i\le|\underline{S}_{n-1}|} \mathsf{H}(\tau_i) \ge (1+\varepsilon)h_n\Big) \le \mathbb{P}\Big(\max_{1\le i\le 2b_n} \mathsf{H}(\tau_i) \ge (1+\varepsilon)h_n\Big) + \mathbb{P}\big(|\underline{S}_{n-1}| > 2b_n\big),$$

which tends to 0 by (3.7) with $c = 2$, since $|\underline{S}_{n-1}| \le |S_{n-1}| + |S_{n-1} - \underline{S}_{n-1}|$ and both $|S_{n-1}|/b_n \to 1$ and $(S_{n-1} - \underline{S}_{n-1})/b_n \to 0$ in probability.

(9) *Proof of Lemma 5.4, pp. 27–29.*

(a) The final step, which absorbs the constant $2b^2C$ into $\lambda^{(r-s)m}$ "by possibly making λ larger", fails when the integer gap $d = (r-s)m$ stays bounded (for instance $d = 1$): a bound $K\lambda^d \le \lambda'^d$ with $K > 1$ and $\lambda' < 1$ requires d large. Lemma 5.4 itself is true. A repair for bounded d: tilt both probabilities by the same parameter $b = b_{c,t}$ with $t = (r+s)/2$. Since G_c is decreasing on $[1, b_{c,\delta}]$, $G_c(b) \le 1$, and the ratio is at most b^{-d} times a ratio of two tilted local probabilities at distance $O(d)$ from the tilted mean, which tends to 1 uniformly by Lemma 5.5; as $b \ge b_{\varepsilon,\delta/4} > 1$, this gives the bound λ^d for any $\lambda \in (b^{-1}_{\varepsilon,\delta/4}, 1)$ and m large. The printed argument covers $d \ge D$ for a suitable constant D.

(b) On p. 27, $b_{c,s}$ is *decreasing*, not increasing, in c when $s < \delta$: at the solution $G'(b_{c,s}) < 0$, and there $bG'_c(b)/G_c(b) = (1-\varepsilon)bG'(b)/((1-\varepsilon)G(b) + c)$ increases

with c. The bounds used remain true: $b_{c,\delta}$ solves $G'(b) = 0$ and does not depend on c, so $1 \le b_{c,s} \le b_{\varepsilon,\delta}$, and $b_{c,(r+s)/2} \ge b_{c,\delta/4} \ge b_{\varepsilon,\delta/4}$.

3. Local mathematical corrections

(10) *Proof of Theorem 1.3, p. 10, line after the two displayed inequalities.* "$(S_{n-1} + b_n)/a_n \to 1$ in probability" is false: by (3.2) this ratio converges in distribution to a non-degenerate limit. Replace the sentence by: "By (3.2), $(S_{n-1} + b_n)/a_n$ is tight; since $a_n = o(b_n)$ and $b_{n-1} \sim b_n$, this implies that $|S_{n-1}|/b_n \to 1$ in probability," which is what is used (the printed $|S_n/b_n|$ should be $|S_{n-1}|/b_n$).

(11) *Proof of Lemma 3.8, p. 12.* The inequality $\sum_{i=0}^{n} \ell(V_\star^{-1}(i)) < \int_0^n \ell(V_\star^{-1}(x))\,dx$ is false: $x \mapsto \ell(V_\star^{-1}(x))$ is decreasing, so a left-endpoint sum dominates the integral. The correct bound is

$$\sum_{i=0}^{n-1} \ell(Q_i) \le \sum_{i=0}^{n-1} \ell(V_\star^{-1}(i)) \le \ell(1) + \int_0^n \ell(V_\star^{-1}(x))\,dx = \ell(1) - \ln V_\star^{-1}(n),$$

and the term $\ell(1) < 1$ is absorbed into the constant C for $n \ge 1$, since $-\ln V_\star^{-1}(n) \ge -\ln V_\star^{-1}(1) > 0$. The statement of Lemma 3.8 is unchanged (for $n \ge 1$).

(12) *Proof of Lemma 3.6, p. 13.* For $0 < \varepsilon < 1$, the equivalence $h_n \sim h'_n$ gives, for all sufficiently large n,

$$(1+\varepsilon)h_n \ge (1+\varepsilon/2)h'_n, \qquad (1-\varepsilon)h_n \le (1-\varepsilon/2)h'_n.$$

In the first display of the proof, replace $1 + 2\varepsilon$ by $1 + \varepsilon/2$ and insert $\le$ between the middle term and the ratio of inverse functions, using $Q_m < V_\star^{-1}(m)$ and $b_n = 1/V_\star^{-1}(h'_n)$. In the second display, replace $1 - 2\varepsilon$ by $1 - \varepsilon/2$ and use h'_n in both terms of the argument of the outer $V_\star^{-1}$ in the numerator. Rapid variation of $V_\star^{-1}$ then gives the two limits.

(13) *Proof of Corollary 4.4, p. 18.* The identity $\max_m \mathcal{V}(\mathbf{s})_m = \max_m s_m - \min_m s_m$ is false; only the inequality $\le$ holds, and it is the only direction used. *Reason.* For the bridge $\mathbf{s} = (0, -1)$, $\mathcal{V}(\mathbf{s}) = (0, -1)$ has maximum 0, while $\max s - \min s = 1$. The inequality holds because every value of $\mathcal{V}(\mathbf{s})$ is a difference $s_j - s_m$ with s_m the minimum (or, after wrapping around, $s_j - s_m - 1$).

(14) *Section 5, overview before Lemma 5.4, p. 24.* Replace "$\sum_{i\ge0} n_i p_i = 1$" by "$\sum_{i\ge0}(n_i - 1)p_i = 0$, that is, $\sum_{i\ge0} n_i p_i = \sum_{i\ge0} p_i$". *Reason.* If $\sum_i n_i p_i = 1$ and the remaining mass $1 - \sum_i p_i > 0$ is placed on 1, the resulting law has mean $1 + 1 - \sum_i p_i > 1$ and is not critical. The construction itself (equation (5.5) and the computation after it) uses the correct identity.