CORRIGENDUM TO "CONDENSATION IN SUBCRITICAL CAUCHY BIENAYMÉ TREES"

ABSTRACT. A formal verification in the Lean proof assistant of the results of [I. Kortchemski and L. Vetter, Condensation in subcritical Cauchy Bienaymé trees, Electron. Commun. Probab. 30 (2025), paper no. 79, 1–13, doi:10.1214/25-ECP726] found the errors listed below (the layout and numbering correspond to the published version).

The statements of the main results, Theorems 1.1 and 3.1, are unaffected; items 6 and 7 repair steps of proofs. The display (2.2), Corollary 4.1 and Proposition 4.5(ii) hold as corrected below, and the claims of Section 2.2 on equivalent laws need the hypothesis $\mu_0 > 0$. The corrected versions of (2.2), Corollary 4.1, Proposition 4.5(ii) and of the classification of Section 2.2 (with $\nu_0 > 0$) are proved in Lean, as are Theorem 3.1, by the argument corrected in item 7, Proposition 4.3, by a variant of the argument of item 6, and the counterexamples of items 1 and 2 except the law ν of item 2. The printed versions of (2.2), Corollary 4.1 and Proposition 4.5(ii) are not refuted in Lean; the reasons given below show that they are false. The corrected criterion of item 1 is checked by hand. The formalization is available at https://github.com/igor-kortchemski/condensation-subcritical-cauchy-bienayme-trees, revision `e112dcf`.

The corrections are grouped into statements and definitions, repairs to proofs, local mathematical corrections, and typographical and bibliographical corrections. Within each section, items follow the order of their first occurrence in the paper.

Throughout, γ_E denotes Euler's constant, and $\mu_0 = \mu(\{0\})$.

1. Corrections to statements and definitions

(1) *Section 2.2, p. 4, paragraph after Proposition 2.1.* The criterion "μ is equivalent to a critical offspring distribution if and only if $\lim_{\lambda\to\rho} \lambda F'_\mu(\lambda)/F_\mu(\lambda) \geq 1$" should read as follows. Set $g(\lambda) = \lambda F'_\mu(\lambda)/F_\mu(\lambda)$, which is nondecreasing on $(0, \rho)$. If $\mu_0 > 0$, then μ is equivalent to a critical offspring distribution if and only if either $\lim_{\lambda\uparrow\rho} g(\lambda) > 1$, or $\lim_{\lambda\uparrow\rho} g(\lambda) = 1$ with $\rho < \infty$ and $F_\mu(\rho) < \infty$ (the law μ_ρ defined by $\mu_\rho(k) = \rho^k \mu_k / F_\mu(\rho)$ is then critical). If $\mu_0 = 0$, then μ is equivalent to a critical offspring distribution only when $\mu = \delta_1$.

Reason. For $\mu_0 = \mu_1 = 1/2$ we have $\rho = \infty$ and $g(\lambda) = \lambda/(1+\lambda) \to 1$, but every law equivalent to μ is supported by $\{0, 1\}$ with positive mass at 0, hence is subcritical.

(2) *Section 2.2, p. 4, last sentence of the same paragraph; the sentence before the list (a)–(e); p. 5, first sentence.* Add the hypothesis $\mu_0 > 0$ to "a supercritical offspring distribution ... is always equivalent to a critical offspring distribution", and the hypothesis $\nu_0 > 0$ to the classification (a)–(e) and to "when ν is supercritical ... we are in case (a)".

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Reason. If $\mu_0 = 0$, every law equivalent to μ has no mass at 0, so its mean is at least 1, with equality only for δ_1. Thus δ_2 is supercritical and not equivalent to a critical law. The law $\nu_0 = 0$, $\nu_n = 6/(\pi^2 n^2)$ for $n \geq 1$, is heavy-tailed with $\alpha = 1$, and every law equivalent to it has mean greater than 1, so none of (a)–(e) applies. Subcritical laws have $\mu_0 > 0$, so the main results are unaffected.

(3) *Section 2.3, p. 5, display (2.2).* The limit is $\mathcal{C}_1 + 1 - \gamma_E$, not $\mathcal{C}_1$:

$$\frac{X_1 + \cdots + X_n - b_n}{a_n} \xrightarrow[n\to\infty]{(d)} \mathcal{C}_1 + 1 - \gamma_E,$$

where $\mathbb{E}[e^{-\lambda \mathcal{C}_1}] = e^{\lambda \log \lambda}$ as in the paper. Equivalently, the limit is $\mathcal{C}_1$ when b_n is replaced by $b_n + (1 - \gamma_E)a_n$.

Reason. The limit is infinitely divisible, with Lévy measure $x^{-2}\,dx$ on $(0, \infty)$, no Gaussian part, and no drift for the truncation $\mathbf{1}_{x \leq 1}$, which matches the centring b_n. Its Laplace exponent is

$$\int_0^\infty \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{x \leq 1}\big) \frac{dx}{x^2} = \lambda \log \lambda + (\gamma_E - 1)\lambda,$$

since the integral equals $\gamma_E - 1$ at $\lambda = 1$. Elsewhere the paper uses (2.2) only through (2.3) and the tightness of $(X_1 + \cdots + X_n - b_n)/a_n$, which a constant shift does not change; only Corollary 4.1 changes (item 4).

(4) *Corollary 4.1, p. 8.* The limit is $-\mathcal{C}_1 - (1 - \gamma_E)$:

$$\frac{\Delta(\mathcal{T}_n) + b_n}{a_n} \xrightarrow[n\to\infty]{(d)} -\mathcal{C}_1 - (1 - \gamma_E).$$

Reason. The paper's derivation from (2.2) and (4.1) is correct given (2.2), and gives this limit once (2.2) is corrected (item 3); replacing n by $n - 1$ in (2.2) changes nothing.

(5) *Proposition 4.5 (ii), p. 11.* Replace $\log(1/m)^{1+\beta}$ by $\log(1/m)^2$: for $\beta \in (0, 1)$ and every $\varepsilon > 0$,

$$\mathbb{P}\big(|\mathsf{H}(\mathcal{T}_n) - h_n| \geq \varepsilon \log(n)^{1-\beta}\big) \xrightarrow[n\to\infty]{} 0,$$

where

$$h_n = \frac{\log(n)}{\log(1/m)} - \frac{c}{(1-\beta)\beta m \log(1/m)^2} \log(n)^{1-\beta}.$$

Reason. The last sentence of the proof (p. 12) correctly obtains $\log(u_n) \sim -c\, n^{1-\beta}/((1-\beta)\beta m \log(1/m)^\beta)$. At $h \sim \log(n)/\log(1/m)$ this gives $\log(u_h) \sim -c \log(n)^{1-\beta}/((1-\beta)\beta m \log(1/m))$, and solving $\log(nQ_h) = 0$ as in the case $\beta = 1$, as the closing clause of that sentence proposes, divides by one more factor $\log(1/m)$ and gives $\log(1/m)^2$. Part (i) is correct: the two coefficients agree for $\beta = 1$. Numerically, iterating (4.5) for $m = 0.5$, $c = 0.3$, $\beta = 0.5$ gives the coefficient 5.00, against 4.995 for the corrected formula and 4.159 for the printed one.

2. Repairs to proofs

(6) *Proof of Proposition 4.3, p. 9.* The reduction "By Theorem 1.1 (ii), it is enough to show (i) and (ii) with $\mathsf{H}(\mathcal{T}_n)$ replaced by $\mathsf{H}_\star(\mathcal{T}_n)$" holds for (i), since $\mathsf{H}(\mathcal{T}_n) \geq \mathsf{H}_\star(\mathcal{T}_n)$, but not for (ii). For (ii), let E_n be the set of vertices of $\mathcal{T}_n$ that are not descendants of a child of $u_\star(\mathcal{T}_n)$. Then

$$\mathsf{H}(\mathcal{T}_n) \leq |E_n| + 1 + \mathsf{H}_\star(\mathcal{T}_n),$$

since E_n contains $u_\star(\mathcal{T}_n)$ and the ancestors of each of its vertices. Moreover $(|E_n|)_{n\geq 1}$ is tight. Indeed, $|E_n| - 1$ is the last zero of the walk whose steps are the increments of the Łukasiewicz path of $\mathcal{T}_n$ other than the maximal one, read cyclically from it in reverse order. This walk depends only on the cyclic order of the increments, so it can be computed from $X_1, \dots, X_n$ under $\mathbb{P}(\,\cdot \mid W_n = -1)$. The maximal increment is unique with probability tending to 1, by Theorem 1.1 (i). Conditionally on this event, the sequence $(X_i^{(n)})$ of Theorem 3.1 is exchangeable and V_n is uniform and independent of it, so these steps have the law of $(X_i^{(n)})$. By Theorem 3.1 they are therefore close in total variation to i.i.d. copies of X, and a random walk with i.i.d. steps of mean $m-1 < 0$ has finitely many nonnegative values. Given $\eta > 0$, choose K with $\mathbb{P}(|E_n| > K) \leq \eta$ for n large. Then $\mathbb{P}(\mathsf{H}(\mathcal{T}_n) \geq h_n) \leq \eta + \mathbb{P}(\mathsf{H}_\star(\mathcal{T}_n) \geq h_n - K - 1)$, and $nQ_{h_n-K-1} \to 0$ because $Q_{j+1}/Q_j \to m > 0$ by Lemma 4.4 (i), so the paper's argument for $\mathsf{H}_\star(\mathcal{T}_n)$ applies with $h_n - K - 1$ in place of h_n.

In the same proof, $[\mathcal{T}_n]_{\lceil \Delta_n/2 \rceil, \Delta_n}$ has the distribution of $[\mathcal{T}_n]_{1,\lfloor \Delta_n/2 \rfloor + 1}$ (reverse the order of the children of $u_\star(\mathcal{T}_n)$), which has one tree more than $[\mathcal{T}_n]_{1,\Delta_n - \lfloor \Delta_n/2 \rfloor}$ when Δ_n is even. The argument is unchanged, since $\mathbb{P}(\lfloor \Delta_n/2 \rfloor + 1 \leq \lfloor \delta n \rfloor) \to 1$ for $\delta = 2(1-m)/3$.

Reason. $\mathsf{H}(\mathcal{T}_n)$ is the maximum of the heights of the vertices of E_n and of the vertices of the forest $[\mathcal{T}_n]_{1,\Delta_n}$. Theorem 1.1 (ii) bounds the height of $u_\star(\mathcal{T}_n)$, hence the second part, but not the heights of the other vertices of E_n.

3. Local mathematical corrections

(7) *Proof of Theorem 3.1, p. 7, line 1.* Replace $x = \gamma n - 1$ by $x = \gamma n - 2$.

Reason. With $\Delta = (0, 1]$, the event $\overline{W}_n \in (\gamma n - 1, \gamma n]$ is $\{W_n = 0\}$, since W_n is an integer; the conditioning event $\{W_n = -1\}$ is $\overline{W}_n \in (\gamma n - 2, \gamma n - 1]$. As $\varepsilon < \gamma$, we still have $x \geq \varepsilon n$ for n large, and the rest of the proof is unchanged.

(8) *p. 10, second line of the first display:* the sum $\sum_{k\geq 1} \sum_{j=0}^{k-1} \mu_j s^k$ should read $\sum_{k\geq 1} \mu_k \sum_{j=0}^{k-1} s^j$ (the final expression is correct).

(9) *p. 11, proof of Proposition 4.5:* "for every $\eta \in (0, 1)$" should read "for every $\eta \in (0, m)$", as in Lemma 4.4 (ii).

4. Typographical and bibliographical corrections

(10) *p. 10, after the first display:* $\ell(0) := \lim_{s\downarrow 0} = 0$ should read $\ell(0) := \lim_{s\downarrow 0} \ell(s) = 0$.

(11) *p. 11, line 1:* $c_n \leq 1/p$ should read $u_n \leq 1/p$.