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Condensation in subcritical Cauchy Bienaymé trees

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Abstract

The goal of this note is to study the geometry of large size-conditioned Bienaymé trees whose offspring distribution is subcritical, belongs to the domain of attraction of a stable law of index $\alpha = 1$ and satisfies a local regularity assumption. We show that a condensation phenomenon occurs: one unique vertex of macroscopic degree emerges, and its height converges in distribution to a geometric random variable. Furthermore, the height of such trees grows logarithmically in their size. Interestingly, the behavior of subcritical Bienaymée trees with $\alpha = 1$ is quite similar to the case $\alpha \in (1, 2]$, in contrast with the critical case. This completes the study of the height of heavy-tailed size-conditioned Bienaymé trees.

Our approach is to check that a random-walk one-big-jump principle due to Armendáriz & Loulakis holds, by using local estimates due to Berger, combined with the previous approach to study subcritical Bienaymé trees with $\alpha > 1$.

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1 Introduction

The main purpose of this work is to complete the study of the height of heavy-tailed size-conditioned Bienaymé trees (sometimes also called Bienaymé–Galton–Watson trees, or Galton–Watson trees in the literature) by considering an offspring distribution $\mu = (\mu_j : j \geq 0)$ such that

$$m := \sum_{k \geq 0} k\mu_k < 1 \qquad \text{and} \qquad \mu_n \underset{n\to\infty}{\sim} \frac{L(n)}{n^2}, \tag{H_μ^{loc}}$$

where $L : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a slowly varying function, meaning that $\lim_{x\to\infty} L(ax)/L(x) = 1$ for all $a > 0$. The first condition amounts to saying that μ is subcritical, while the second condition implies that the offspring distribution is in the domain of attraction of a stable law of index $\alpha = 1$ (i.e. a Cauchy distribution).

We denote by $\mathcal{T}_n$ a μ-Bienaymé tree conditioned to have n vertices (we always implicitly restrict to those values of n for which this event has non-zero probability). We

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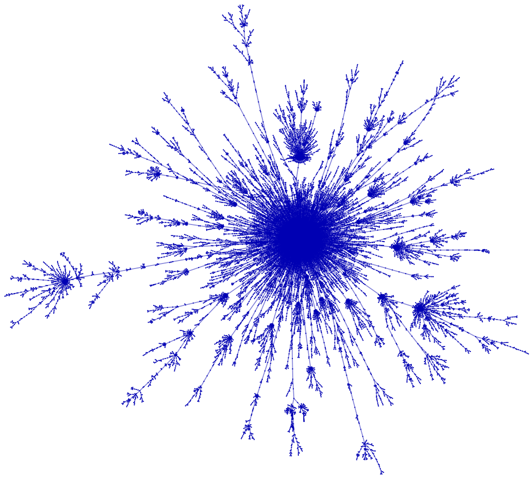

Figure 1: A simulation of a subcritical Cauchy Bienaymé tree with 20000 vertices.

let $\Delta(\mathcal{T}_n)$ be its maximum outdegree, $\Delta^2(\mathcal{T}_n)$ be its second maximal outdegree, $\mathsf{H}_\Delta(\mathcal{T}_n)$ be the height of the vertex with maximal outdegree (first in lexicographical order if many, see below for details) and finally we let $\mathsf{H}(\mathcal{T}_n)$ be the height of $\mathcal{T}_n$, i.e. the maximal graph distance of the root to any of its vertices.

Theorem 1.1. *The following assertions hold.*

(i) [condensation] We have $\dfrac{\Delta(\mathcal{T}_n)}{n(1-m)} \xrightarrow[n\to\infty]{(\mathbb{P})} 1$ *and* $\dfrac{\Delta^2(\mathcal{T}_n)}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} 0.$

(ii) [height of condensation vertex] For every $j \geq 0$ *we have*

$$\mathbb{P}(H_\Delta(\mathcal{T}_n) = j) \xrightarrow[n\to\infty]{} (1-m)m^j.$$

(iii) [height of the tree]

(a) The following convergence holds in probability

$$\frac{\mathsf{H}(\mathcal{T}_n)}{\log(n)} \xrightarrow[n\to\infty]{(\mathbb{P})} \frac{1}{\log(1/m)}.$$

(b) The sequence $\left(\mathsf{H}(\mathcal{T}_n) - \frac{\log(n)}{\log(1/m)}\right)_{n\geq 1}$ *is tight if and only if* $\sum_{n\geq 1}(n\log n)\mu_n < \infty$*.*

This shows in particular that a condensation phenomenon occurs, in the sense that the maximal degree of $\mathcal{T}_n$ is comparable to the total size of the tree while the second largest degree is negligible compared to the total size of the tree. See Corollary 4.1 for the fluctuations of $\Delta(\mathcal{T}_n)$. We discuss some aspects of Theorem 1.1 after describing the context.

1.1 Context

The condensation phenomenon was discovered by Jonsson and Stefánsson [20] for subcritical offspring distributions satisfying $\mu_n \sim c/n^{1+\alpha}$ as $n \to \infty$ with $\alpha > 1$ and $c > 0$ (see also [18, Sec. 19.6]). This was extended in [22] to subcritical offspring distributions satisfying $\mu_n \sim L(n)/n^{1+\alpha}$ with L slowly varying and $\alpha > 1$, where the same conclusions as those of Theorem 1.1 were established.

In the critical case, when $\alpha > 1$ there is no condensation phenomenon occurring. Indeed, for critical offspring distributions, when μ has finite variance (which requires $\alpha \geq 2$), the scaling limit of $\mathcal{T}_n$ is Aldous' Brownian Continuum Random Tree [5], and when μ has infinite variance (which requires $\alpha \in (1, 2]$), the scaling limit of $\mathcal{T}_n$ is the so-called the α-stable tree [13].

There is no condensation phenomenon neither for supercritical offspring distributions. Indeed, the study of size-conditioned Bienaymé trees with supercritical offspring distribution is equivalent to the study of size-conditioned Bienamyé trees with critical finite variance offsprings by exponential tilting (see Sec. 2.2 below), which are covered by Aldous' previously mentioned result.

The so-called Cauchy case where $\alpha = 1$ has only been considered quite recently in the critical case [23], motivated by applications to random planar maps: a condensation phenomenon occurs, although on a slightly different scale (the unique vertex of maximal degree is of order $o(n)$, yet dominates the degrees of the other vertices). Limit theorems for the height of such trees have been established in [2].

In the recent years, it has been realized that Bienaymé trees in which a condensation phenomenon occurs code a variety of random combinatorial structures such as random planar maps [1, 19, 25, 4, 3], outerplanar maps [26], supercritical percolation clusters of random triangulations [11], random permutations [9], parking on random trees [10] or minimal factorizations [15]. See [27] for a combinatorial framework and further examples. These applications are one of the motivations for the study of the fine structure of such large conditioned Bienaymé trees.

1.2 Comments on Theorem 1.1

The goal of this paper is to cover the case where μ is subcritical and $\alpha = 1$, which is the last missing case for heavy-tailed offspring distributions satisfying a local regularity assumption (see Sec. 2.2 below). The results of Theorem 1.1 (i), (ii) and (iii) (a) are the same as for subcritical offspring distributions with $\alpha > 1$ [22]. However, an interesting new phenomenon appears in the case $\alpha = 1$: indeed, the sequence $\left(\mathsf{H}(\mathcal{T}_n) - \log(n)/\log(1/m)\right)_{n\geq 1}$ is not always tight (in contrast with the case $\alpha > 1$ where it is always tight). The reason is that when $\alpha > 1$ we always have $\sum_{n\geq 1}(n \log n)\mu_n < \infty$, but in the case $\alpha = 1$ we may have $\sum_{n\geq 1}(n \log n)\mu_n = \infty$; take for example $\mu_n \sim \frac{c}{(\log n)^{1+\beta} n^2}$ with $\beta \in (0, 1]$. See Proposition 4.5 for the second order term for the magnitude of $\mathsf{H}(\mathcal{T}_n)$ in this particular case.

It is also interesting to note that for subcritical offspring distributions, the behavior of large size-conditioned Bienaymé trees turns out to be similar for $\alpha = 1$ and $\alpha \in (1, 2]$, while for critical offspring distributions, the behavior of large size-conditioned Bienaymé trees for $\alpha = 1$ and $\alpha \in (1, 2]$ is quite different (there are no non-trivial scaling limits in the case $\alpha = 1$).

1.3 Main ideas

Using [7], we show that the one-big jump principle of [6] used in the case $\alpha > 1$ in [22] holds when $\alpha = 1$ as well. Theorem 1.1 (i) and (ii) then follow as in [22] from this one-big jump principle. However the proof of Theorem 1.1 (iii) concerning the height of $\mathcal{T}_n$ requires different estimates. Indeed, the estimate $\mathbb{P}(\mathsf{H}(\mathcal{T}) \geq n) \sim c \cdot m^n$, where $\mathcal{T}$ is an unconditioned μ-Bienaymé tree, used in [22] for $\alpha > 1$, is not true in general for $\alpha = 1$.

1.4 Outline

We first introduce Bienaymé trees and their associated random walks in Section 2. The main one big jump principle for conditioned random walks is presented in Section 3, and limit theorems for subcritical Cauchy Bienaymé trees are proved in Section 4.

2 Bienaymé trees and random walks

2.1 Bienaymé trees

We consider plane trees, which are also sometimes called rooted ordered trees (see [16, Sec. 1] for definitions and background). For every plane tree τ and every vertex $v \in \tau$, we denote by $k_v(\tau)$ the outdegree (or number of children) of v, so that $\Delta(\tau) = \max_{v \in \tau} k_v(\tau)$ is the maximal outdegree of τ.

Given a probability distribution $\mu = (\mu_n)_{n \geq 0}$, on the nonnegative integers $\mathbb{Z}_+$, we denote by $\mathbb{P}_\mu$ the law of a Bienaymé tree with offspring distribution μ. It satisfies for every finite tree τ the identity

$$\mathbb{P}_\mu(\tau) = \prod_{u \in \tau} \mu_{k_u(\tau)}.$$

For every $n \geq 1$, we denote by $\mathcal{T}_n^\mu$ a Bienaymé tree with offspring distribution μ conditioned to have n vertices (we always implicitly restrict to those values of n for which this event has non-zero probability).

2.2 Exponential tilting

Exponential tilting is a useful tool which allows to tune the mean of a Bienaymé tree without changing the law of the associated size-conditioned Bienaymé tree. It is essentially due to Kennedy [21] (in the context of branching processes).

Proposition 2.1 (Kennedy)**.** *Let μ and ν be two probability distributions on $\mathbb{Z}_+$ such that for certain $a, \lambda > 0$ we have $\mu_k = a\lambda^k \nu_k$ for every $k \geq 0$. Then $\mathcal{T}_n^\mu$ and $\mathcal{T}_n^\nu$ have the same distribution.*

If μ and ν are related as in Proposition 2.1, we say that they are equivalent. It is a simple matter to characterize the class of offspring distributions μ which are equivalent to a critical offspring distribution. Indeed, let $F_\mu(x) = \sum_{k=0}^\infty \mu_k x^k$ be the generating function of μ and denote by ρ its radius of convergence. Then for every $\lambda \in (0, \rho)$, the offspring distribution μ_λ with generating function $F_{\mu_\lambda}(x) = F_\mu(\lambda x)/F_\mu(\lambda)$ is equivalent to μ and has expectation $\lambda F'_\mu(\lambda)/F_\mu(\lambda)$. Thus μ is equivalent to a critical offspring distribution if and only if $\lim_{\lambda \to \rho} \lambda F'_\mu(\lambda)/F_\mu(\lambda) \geq 1$. In particular, observe that a supercritical offspring distribution (i.e. that has mean greater than 1) is always equivalent to a critical offspring distribution.

As a consequence, if ν is a heavy-tailed offspring distribution of the form $\nu_n \sim L(n)/n^{1+\alpha}$ with $\alpha \geq 0$ and L slowly varying then ν is equivalent to one of the following:

(a) a critical offspring distribution μ with finite variance. In this case the asymptotic behavior of $\mathcal{T}_n^\mu$ is known [5].

(b) a critical offspring distribution μ with infinite variance and $\alpha \in (1, 2]$. In this case the asymptotic behavior of $\mathcal{T}_n^\mu$ is known [13].

(c) a critical offspring distribution μ with $\alpha = 1$. In this case the asymptotic behavior of $\mathcal{T}_n^\mu$ is known [2].

(d) a subcritical offspring distribution with $\alpha > 1$. In this case the asymptotic behavior of $\mathcal{T}_n^\mu$ is known [22].

(e) a subcritical offspring distribution μ with $\alpha = 1$. This case is last missing case considered in this note.

Observe that when ν is supercritical (which includes the case $\alpha < 1$ since ν has then infinite mean) we are in case (a).

2.3 Random walks

An important tool to study Bienaymé trees is the use of random walks. Given an offspring distribution μ, let X be a random variable with law given by $\mathbb{P}(X = i) = \mu_{i+1}$ for $i \in \mathbb{Z}_{\geq -1}$. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables distributed as X and let $(W_k)_{k\geq 0}$ be the random walk defined by $W_0 = 0$ and $W_k = \sum_{i=1}^{k} X_i$ for $k \geq 1$.

The key connection between Bienaymé trees and random walks is that is it possible to bijectively code plane trees by the so-called Łukasiewicz path, in such a way that the Łukasiewicz path of a Bienaymé tree has the law of $(W_k)_{k\geq 0}$ stopped at the first hitting time of the negative integers. As a consequence, studying $\mathcal{T}_n^{\mu}$ is equivalent to studying the random walk $(W_k)_{k\geq 0}$ conditioned on hitting the negative integers at time n ("excursion"-type conditioning). In turn, this is equivalent to studying the random walk $(W_k)_{k\geq 0}$ conditioned on hitting -1 at time n ("bridge"-type conditioning) using the Vervaat transform. One of the key implications is that the collection of outdegrees minus 1 of $\mathcal{T}_n$ has the same law as the jumps of $(W_k)_{0\leq k\leq n}$ under the conditional probability $\mathbb{P}(\,\cdot \mid W_n = -1)$, see [24, Sec. 6.1] for background.

From now on, assume that μ satisfies (H_{μ}^{loc}). We need to introduce two sequences related to the asymptotic behavior of $(W_n)_{n\geq 1}$. Observe that $\mathbb{E}[X] = m - 1 < 0$. Let $(a_n : n \geq 1)$ and $(b_n : n \geq 1)$ be sequences such that

$$n\mathbb{P}(X \geq a_n) \xrightarrow[n\to\infty]{} 1, \qquad b_n = n\mathbb{E}[X\mathbb{1}_{|X|\leq a_n}], \tag{2.1}$$

Then the following convergence holds in distribution

$$\frac{X_1 + \cdots + X_n - b_n}{a_n} \xrightarrow[n\to\infty]{(d)} \mathcal{C}_1, \tag{2.2}$$

where $\mathcal{C}_1$ is a random variable, with Laplace transform given by $\mathbb{E}[e^{-\lambda \mathcal{C}_1}] = e^{\lambda \log \lambda}$ for $\lambda > 0$, see [14, Chap. IX.8 and Eq. (8.15) p.315]. For this reason, we often call $(a_n : n \geq 1)$ the scaling sequence and $(b_n : n \geq 1)$ the centering sequence. The random variable $\mathcal{C}_1$ is an asymmetric Cauchy random variable with skewness 1. In addition (a_n) and (b_n) are regularly varying sequences of index 1.

We will also need to introduce an auxiliary slowly varying function. For every $n \geq 1$ set $\ell^\star(n) \coloneqq \sum_{k=n}^{\infty} L(k)/k$ for $n \geq 1$, which is a finite quantity since μ has finite mean.

Lemma 2.2. *The following assertions hold as $n \to \infty$.*

- *(i) The function $\ell^\star$ is slowly varying, satisfies $\ell^\star(n) \to 0$ and $L(n) = o(\ell^\star(n))$.*
- *(ii) We have $b_n + n(1 - m) \sim -n\ell^\star(a_n)$ and $b_n \sim -n(1 - m)$.*
- *(iii) We have $a_n = o(n)$.*

Proof. For the first assertion, by definition we clearly have $\ell^\star(n) \to 0$ as $n \to \infty$. The two other properties follow from [8, Proposition 1.5.9b].

For (ii), write $b_n = n(\mathbb{E}[X] - \mathbb{E}[X\mathbb{1}_{|X|>a_n}]) = n(m-1) - n\mathbb{E}[X\mathbb{1}_{|X|>a_n}]$ and observe that by assumption (H_{μ}^{loc}) we have $n\mathbb{E}[X\mathbb{1}_{|X|>a_n}] \sim n\ell^\star(a_n)$, which gives the first asymptotic estimate. The second one follows from the fact that $\ell^\star(a_n) \to 0$ because $a_n \to \infty$.

Finally, for the last assertion, we combine the fact that $a_n \sim nL(a_n)$ (by definition of a_n) with the fact that $\ell^\star(n) \to 0$ and write for n sufficiently large

$$\frac{a_n}{n} \leq \frac{nL(a_n)}{n\ell^\star(a_n)},$$

which converges to 0 since $L(n) = o(\ell^\star(n))$. □

In particular, (2.2) and Lemma 2.2 imply that

$$\frac{X_1 + \cdots + X_n}{n(m-1)} \xrightarrow[n\to\infty]{(\mathbb{P})} 1. \tag{2.3}$$

Also observe that by Lemma 2.2 (i) and (ii), it is not true that $(W_n - \mathbb{E}[W_n])/a_n)$ converges in distribution as $n \to \infty$: in contrast with the case $\alpha > 1$ some care is needed in handling the centering term.

Example 2.3. In some particular cases we can explicitly determine the asymptotic behavior of a_n and $\ell^\star$:

(i) If $L(n) \sim c/\log(n)^{1+\beta}$ with $c, \beta > 0$, then $\ell^\star(n) \sim c/(\beta \log(n)^\beta)$ and $a_n \sim nL(n)$ since $\log(a_n) \sim \log(n)$.

(ii) Set $\log_{(1)}(x) = \log(x)$ and for every $k \geq 1$ define recursively $\log_{(k+1)}(x) = \log(\log_{(k)}(x))$. For

$$L(n) \quad \sim \quad \frac{c}{(\log_{(k)}(n))^2} \prod_{i=1}^{k-1} \frac{1}{\log_{(i)}(n)}$$

with $k \geq 2$ and $c > 0$, we have $\ell^\star(n) \sim c/\log_{(k)}(n)$ and $a_n \sim nL(n)$ since $\log(a_n) \sim \log(n)$.

(iii) For $L(n) \sim c\exp(-\log(n)^\beta)$ with $\beta \in (0,1)$ and $c > 0$, we have $\ell^\star(n) \sim c/\beta \cdot \log(n)^{1-\beta} \exp(-\log(n)^\beta)$. The asymptotic behavior of a_n depends on the value of β, for example when $\beta < 1/2$ we have $a_n \sim cn\exp(-\log(n)^\beta)$ and when $1/2 \leq \beta < 2/3$ we have $a_n \sim cn\exp(-\log(n)^\beta + \beta \log(n)^{2\beta-1})$.

3 A one big jump principle for conditioned random walks

Here we consider an offspring distribution μ satisfying (H_μ^{loc}) and denote by $(W_k)_{k\geq 0}$ the random walk defined in Sec. 2.3. The key ingredient that enables us to use the results of [22] is a one-big jump principle for the random walk under the conditional probability $\mathbb{P}(\cdot \mid W_n = -1)$.

In the case $\alpha > 1$, this was obtained in [22] thanks to a general result due to Armendáriz & Loulakis [6], using local estimates for random walks obtained in [12]. In the case $\alpha = 1$, we show that we can still use the result of Armendáriz & Loulakis [6], using instead local estimates for random walks obtained by Berger [7].

For every integer $n \geq 1$, let V_n be the index of the first maximal jump of $(W_k)_{0\leq k\leq n}$ defined by

$$V_n := \inf\Big\{1 \leq j \leq n : X_j = \max\{X_i : 1 \leq i \leq n\}\Big\}.$$

Then we define $(X_1^{(n)}, ..., X_{n-1}^{(n)})$ to be the random variable distributed as the law of $(X_1, ..., X_{V_n-1}, X_{V_n+1}, ..., X_n)$ under $\mathbb{P}(\cdot \mid W_n = -1)$. The following theorem states that once the first maximal jump of $(W_k)_{0\leq k\leq n}$ under $\mathbb{P}(\cdot \mid W_n = -1)$ is removed, the remaining increments behave asymptotically like i.i.d. random variables.

Theorem 3.1. *We have*

$$d_{\mathrm{TV}}\left((X_i^{(n)} : 1 \leq i \leq n-1), (X_i : 1 \leq i \leq n-1)\right) \xrightarrow[n\to\infty]{} 0$$

where d_{TV} denotes the total variation distance on $\mathbb{R}^{n-1}$.

Proof. To simplify notation, set $\gamma = 1 - m$. For every $n \geq 1$, set $\overline{X}_n = X_n + \gamma$ and $\overline{W}_n = W_n + \gamma n$, so that $(\overline{W}_n)_{n\geq 0}$ is a centered random walk. Similarly, set $(\overline{X}_1^{(n)}, \ldots, \overline{X}_{n-1}^{(n)}) = (X_1^{(n)} + \gamma, \ldots, X_{n-1}^{(n)} + \gamma)$. Fix $\varepsilon \in (0, \gamma)$. We check that we can apply Theorem 1 in [6], with μ being the law of $\overline{X}_1$, $\Delta = (0,1]$, $q_n = \varepsilon n$ and $x = \gamma n - 1$. This will indeed imply that

$$d_{\mathrm{TV}}\left((\overline{X}_i^{(n)} : 1 \leq i \leq n-1), (\overline{X}_i : 1 \leq i \leq n-1) \right) \xrightarrow[n\to\infty]{} 0,$$

giving the desired result.

In order to apply Theorem 1 in [6], we check that condition (2.6) there holds with $d_n = \varepsilon n$ and that condition (3.3) there holds also with $\ell_n = \varepsilon n$. Recall the definition of $\ell^\star$ introduced just before Lemma 2.2.

Condition (2.6). We need to check that

$$\lim_{n\to\infty} \sup_{x \geq \varepsilon n} \left| \frac{\mathbb{P}(\overline{W}_n \in (x, x+1])}{n\mathbb{P}(\overline{X}_1 \in (x, x+1])} - 1 \right| = 0,$$

or, equivalently, setting $m_n = b_n + \gamma n$,

$$\lim_{n\to\infty} \sup_{x \geq \varepsilon n} \left| \frac{\mathbb{P}(W_n - b_n \in (x - m_n, x - m_n + 1])}{n\mathbb{P}(X_1 \in (x-\gamma, x-\gamma+1])} - 1 \right| = 0. \tag{3.1}$$

We claim that $|x - m_n|/a_n \to \infty$ uniformly in $x \geq \varepsilon n$. Indeed, by Lemma 2.2 (ii), we have $m_n \sim -n\ell^\star(a_n)$, so that $|m_n| = o(n)$ since $\ell^\star(n) \to 0$ as $n \to \infty$. It follows that $|x - m_n| \sim x$ uniformly in $x \geq \varepsilon n$, and we get our claim since $a_n = o(n)$ (Lemma 2.2 (iii)). This puts us in position to use Theorem 2.4 in [7] (in the reference we take $\alpha = 1$ and $x = -\lfloor b_n \rfloor - 1$), which gives that

$$\lim_{n\to\infty} \sup_{x \geq \varepsilon n} \left| \frac{\mathbb{P}(W_n - b_n \in (x - m_n, x - m_n + 1])}{n\mathbb{P}(X_1 \in (x - m_n, x - m_n + 1])} - 1 \right| = 0. \tag{3.2}$$

Now observe that since L is slowly varying at infinity, for any sequence $\delta_n \to 0$ of positive real numbers and any sequence of integers $z_n \to \infty$ we have the convergence $\sup_{(1-\delta_n)z_n \leq y \leq (1+\delta_n)z_n} L(y)/L(z_n) \to 1$ (this follows e.g. from the representation theorem for slowly varying functions). Using (H_μ^{loc}) and $|x - m_n| \sim x$ uniformly in $x \geq \epsilon n$ together with the assumption that $L(\cdot)$ is slowly varying at infinity, we get

$$\lim_{n\to\infty} \sup_{x \geq \varepsilon n} \left| \frac{\mathbb{P}(X_1 \in (x - m_n, x - m_n + 1])}{\mathbb{P}(X_1 \in (x-\gamma, x-\gamma+1])} - 1 \right| = 0.$$

Combined with (3.2) we get (3.1).

Condition (3.3). Set $\widehat{b}_n = n\ell^\star(a_n)$. We check that $(\overline{W}_n/\widehat{b}_n)_{n\geq 1}$ is tight and that for every $L > 0$ we have

$$\sup_{x \geq \varepsilon n} \sup_{|y| \leq L\widehat{b}_n} \left| 1 - \frac{\mathbb{P}(\overline{X}_1 \in (x-y, x-y+1])}{\mathbb{P}(\overline{X}_1 \in (x, x+1])} \right| \xrightarrow[n\to\infty]{} 0. \tag{3.3}$$

To check tightness, write

$$\frac{\overline{W}_n}{\widehat{b}_n} = \frac{X_1 + \cdots + X_n - b_n}{a_n} \cdot \frac{a_n}{\widehat{b}_n} + \frac{b_n + \gamma n}{\widehat{b}_n},$$

which implies tightness since $a_n/\widehat{b}_n \sim L(a_n)/\ell^\star(a_n) \to 0$ and $(b_n + \gamma n)/\widehat{b}_n \to -1$ by Lemma 2.2 (ii). The convergence (3.3) readily follows from the fact that $\widehat{b}_n = o(n)$ and the fact that for every $\delta > 0$ and for every positive sequence (η_n) of real numbers such that $\eta_n = o(n)$ we have

$$\sup_{x \geq \varepsilon n} \sup_{|u| \leq \eta_n} \left| \frac{L(x+u)}{L(x)} - 1 \right| \xrightarrow[n\to\infty]{} 0.$$

This can e.g. be seen using the representation theorem for slowly varying functions [8, Theorem 1.3.1]. This completes the proof. □

This establishes the same one-big jump principle as when μ is subcritical and $\mu(n) \sim L(n)/n^{1+\alpha}$ with $\alpha > 1$, which is Theorem 2.1 in [22].

4 Condensation in subcritical Cauchy Bienaymé trees

We are now ready to establish our results for subcritical Cauchy Bienaymé trees. As before, we consider an offspring distribution μ satisfying (H_μ^{loc}) and denote by $(W_k)_{k\geq 0}$ the random walk defined in Sec. 2.3. Recall that $\mathcal{T}_n$ denotes a Bienaymé tree with offspring distribution μ conditioned to have n vertices.

4.1 Condensation phenomenon

Theorem 1.1 (i) and (ii) are proved in the same way as Theorem 1 and Theorem 2 are proved in [22] in the case where μ is subcritical and $\mu(n) \sim L(n)/n^{1+\alpha}$ with $\alpha > 1$. Indeed, for Theorem 1.1 (i), by combining the one big jump principle (Theorem 3.1) with the fact that the collection of outdegrees minus 1 of $\mathcal{T}_n$ has the same law as the collection of jumps of $(W_k)_{0\leq k\leq n}$ under the conditional probability $\mathbb{P}(\,\cdot \mid W_n = -1)$, we get

$$\mathrm{d}_{\mathrm{TV}}\left(\Delta(\mathcal{T}_n), -(X_1 + \cdots + X_{n-1})\right) \xrightarrow[n\to\infty]{} 0, \tag{4.1}$$

which by (2.3) yields the first convergence of Theorem 1.1 (i). Also,

$$\mathrm{d}_{\mathrm{TV}}\left(\Delta^2(\mathcal{T}_n) - 1, \max(X_1, \ldots, X_{n-1})\right) \xrightarrow[n\to\infty]{} 0,$$

which using the fact that $\mathbb{P}(X_1 \geq u) \sim L(u)/u$ as $u \to \infty$ implies that $\max(X_1, \ldots, X_{n-1})/a_n$ converges in distribution to a random variable Y with law given by $\mathbb{P}(Y \leq u) = \exp(-1/u)$ for $u > 0$. Since $a_n = o(n)$ this implies the second convergence of Theorem 1.1 (i).

Theorem 1.1 (ii) is established in the exact same way Theorem 2 in [22] is proved, taking as input the one big jump principle (Theorem 3.1 in our case $\alpha = 1$).

Also, by combining (2.2) with (4.1) we immediately get the following fluctuations for $\Delta(\mathcal{T}_n)$:

Corollary 4.1. *We have*

$$\frac{\Delta(\mathcal{T}_n) + b_n}{a_n} \xrightarrow[n\to\infty]{(d)} -\mathcal{C}_1.$$

As suggested by an anonymous referee, it would be very interesting to extend this central limit theorem to a local limit theorem, motivated by Stufler's result [28, Lemma 2.2] for $\alpha > 1$ (the proof of [28, Lemma 2.2] strongly relies on the fact that $\alpha > 1$, so new input is needed).

4.2 Height of $\mathcal{T}_n$

In [22], the proof of the fact that $\mathsf{H}(\mathcal{T}_n)/\log(n) \to 1/\log(1/m)$ in probability when μ is subcritical and $\mu(n) \sim L(n)/n^{1+\alpha}$ with $\alpha > 1$ uses the asymptotic estimate $\mathbb{P}(\mathsf{H}(\mathcal{T}) \geq n) \sim c \cdot m^n$, where $\mathcal{T}$ is an unconditioned μ-Bienaymé tree. However, this estimate is not true in general when $\alpha = 1$. Indeed, by [17, Theorem 2], this asymptotic estimate holds if and only if $\sum_{n\geq 1}(n \log n)\mu_n < \infty$; when this sum is infinite we have $\mathbb{P}(\mathsf{H}(\mathcal{T}) \geq n)/m^n \to 0$ as $n \to \infty$. Observe that in the case $\alpha = 1$, as was already mentioned we can have $\sum_{n\geq 1}(n \log n)\mu_n = \infty$.

In the case $\alpha = 1$, we use the same idea as in [22], which consists in using the fact that, roughly speaking, the trees grafted on the vertex of maximal degree of $\mathcal{T}_n$ are asymptotically independent μ-Bienaymé trees, combined with a bound on $\mathbb{P}(\mathsf{H}(\mathcal{T}) \geq n)$.

We need to introduce some notation. For every finite plane tree τ, denote by $u_\star(\tau)$ the vertex of maximal degree (first in lexicographical order if not unique) and for every

$1 \le i \le \Delta(\tau)$, let τ_i be the tree of descendants of the i-th child of $u_\star(\tau)$. For $i > \Delta(\tau)$ we set $\tau_i = \varnothing$. For $1 \le j \le k$, we let $[\tau]_{j,k}$ be the forest defined by $[\tau]_{j,k} = \{\tau_i : j \le i \le k\}$. Finally for every $i \ge 1$, denote by $\mathcal{F}^i$ the forest of i independent μ-Bienaymé trees.

Proposition 4.2. *For every $\delta \in [0, 1-m)$, we have $\mathrm{d_{TV}}\left([\mathcal{T}_n]_{1,\lfloor \delta n \rfloor}, \mathcal{F}^{\lfloor \delta n \rfloor}\right) \to 0$ as $n \to \infty$.*

This is established in the exact same way Corollary 2.7 is proved in [22], again taking as input the the one big jump principle (Theorem 3.1 in our case $\alpha = 1$).

For every $n \ge 0$ set $Q_n = \mathbb{P}(H(\mathcal{T}) \ge n)$, where $\mathcal{T}$ is an (unconditioned) μ-Bienaymé tree.

Proposition 4.3. *Let (h_n) be a sequence of positive real numbers such that $h_n \to \infty$. The following assertions hold.*

(i) If $nQ_{h_n} \to \infty$ then $\mathbb{P}(\mathsf{H}(\mathcal{T}_n) \ge h_n) \to 1$ as $n \to \infty$.
(ii) If $nQ_{h_n} \to 0$ then $\mathbb{P}(\mathsf{H}(\mathcal{T}_n) < h_n) \to 1$ as $n \to \infty$.

Proof. We mimic the proof of Theorem 4 in [22]. For every tree τ, denote by $\mathsf{H}_\star(\tau)$ the height of the forest $[\tau]_{1,\Delta(\tau)}$. To simplify notation, we set $\Delta_n = \Delta(\mathcal{T}_n)$. By Theorem 1.1 (ii), it is enough to show (i) and (ii) with $\mathsf{H}(\mathcal{T}_n)$ replaced by $\mathsf{H}_\star(\mathcal{T}_n)$.

We start with (i). Observe that by Theorem 1.1 (i), setting $\delta = (1-m)/2$, we have $\mathbb{P}(\Delta_n \ge \lfloor \delta n \rfloor) \to 1$ as $n \to \infty$. Thus, using Proposition 4.2,

$$\mathbb{P}(\mathsf{H}_\star(\mathcal{T}_n) < h_n) \le \mathbb{P}(\mathsf{H}([\mathcal{T}_n]_{1,\lfloor \delta n \rfloor}) < h_n) + o(1) = \mathbb{P}(\mathsf{H}(\mathcal{F}^{\lfloor \delta n \rfloor}) < h_n) + o(1).$$

But $\mathbb{P}(\mathsf{H}(\mathcal{F}^{\lfloor \delta n \rfloor}) < h_n) = (1 - Q_{h_n})^{\lfloor \delta n \rfloor} \to 0$ since $nQ_{h_n} \to \infty$.

For (ii), write

$$\mathbb{P}(\mathsf{H}_\star(\mathcal{T}_n) \ge h_n) \le \mathbb{P}(\mathsf{H}_\star([\mathcal{T}_n]_{1,\lfloor \Delta_n/2 \rfloor}) \ge h_n) + \mathbb{P}(\mathsf{H}_\star([\mathcal{T}_n]_{\lceil \Delta_n/2 \rceil, \Delta_n}) \ge h_n).$$

Since $[\mathcal{T}_n]_{\lceil \Delta_n/2 \rceil, \Delta_n}$ and $[\mathcal{T}_n]_{1, \Delta_n - \lfloor \Delta_n/2 \rfloor}$ have the same distribution, it suffices to show that $\mathbb{P}(\mathsf{H}_\star([\mathcal{T}_n]_{1,\lfloor \Delta_n/2 \rfloor}) \ge h_n) \to 0$ as $n \to \infty$. Set $\delta = 2(1-m)/3$. By Theorem 1.1 (i), we have $\mathbb{P}(\lfloor \Delta_n/2 \rfloor \le \lfloor \delta n \rfloor) \to 1$ as $n \to \infty$. Thus, using Proposition 4.2,

$$\mathbb{P}(\mathsf{H}_\star([\mathcal{T}_n]_{1,\lfloor \Delta_n/2 \rfloor}) \ge h_n) \le \mathbb{P}(\mathsf{H}([\mathcal{T}_n]_{1,\lfloor \delta n \rfloor} \ge h_n) + o(1) = \mathbb{P}(\mathsf{H}(\mathcal{F}^{\lfloor \delta n \rfloor}) \ge h_n) + o(1).$$

But $\mathbb{P}(\mathsf{H}(\mathcal{F}^{\lfloor \delta n \rfloor}) \ge h_n) = 1 - \mathbb{P}(\mathsf{H}(\mathcal{F}^{\lfloor \delta n \rfloor}) < h_n) = 1 - (1 - Q_{h_n})^{\lfloor \delta n \rfloor} \to 0$ since $nQ_{h_n} \to 0$. This completes the proof. ☐

In order to apply Proposition 4.3 we will use the following bounds on Q_n.

Lemma 4.4. *The following assertions hold.*

(i) There is a function $\ell : (0,1] \to \mathbb{R}_+^*$ slowly varying at 0 such that $\ell(x) \to 0$ as $x \to 0$ and $Q_{n+1} = Q_n\big(m - \ell(Q_n)\big)$ for every $n \ge 0$.
(ii) For every $\eta \in (0, m)$, for every n sufficiently large we have $(m-\eta)^n \le Q_n \le m^n$.

Proof. Let $G_\mu(t) = \sum_{k \ge 0} t^k \mu_k$ be the probability generating function of μ. Since μ satisfies (H_μ^{loc}), we may apply Karamata's Abelian theorem [8, Theorem 8.1.6] (in the reference we take $n = \alpha = 1, \beta = 0$, $f_1(s) = G_\mu(e^{-s}) - 1 + ms$, substitute s by $-\log(s)$ and Taylor expand the logarithm) to write for $s \in (0,1]$

$$G_\mu(s) = 1 - m(1-s) + (1-s)\ell(1-s) \tag{4.2}$$

for a function $\ell : (0,1] \to \mathbb{R}$ with $\ell(x) \sim \ell^\star(1/x)$ as $x \to 0$. We claim that for $s \in (0,1]$ we have

$$\ell(s) = m - \sum_{k \ge 0} \mu([k+1, \infty))(1-s)^k. \tag{4.3}$$

This follows from (4.2) by writing $\ell(1-s)$ as

$$\begin{aligned} m+\frac{G_\mu(s)-1}{1-s} &= m+\frac{1}{1-s}\sum_{k\geq 0}\mu_k(s^k-1) = m-\frac{1}{1-s}\sum_{k\geq 1}\mu_k(1-s)(s^{k-1}+\cdots+1) \\ &= m-\sum_{k\geq 1}\sum_{j=0}^{k-1}\mu_j s^k = m-\sum_{k\geq 0}\mu([k+1,\infty))s^k. \end{aligned}$$

From (4.3) it follows that $\ell(0) := \lim_{s\downarrow 0} = 0$, $\ell(1) = m-1+\mu_0 < m$ and $\ell(s)$ is increasing. Thus, $0 < \ell(s) < m$ for every $s \in (0,1]$.

Now, by decomposing $\mathcal{T}$ into the forest of subtrees rooted at children of the root we get

$$Q_{n+1} = \sum_{k\geq 1}\mu_k\Big(1-\mathbb{P}(\mathsf{H}(\mathcal{T})<n)^k\Big) = \sum_{k\geq 1}\mu_k\Big(1-(1-Q_n)^k\Big) = 1-G_\mu(1-Q_n).$$

By substituting the expression of G_μ given by (4.2), we get (i).

We turn to (ii). The upper bound simply comes from the fact that ℓ is positive, implying that $Q_{n+1} \leq mQ_n$ for every $n \geq 0$. For the lower bound, fix $\eta \in (0,m)$. Since $\ell(x) \to 0$ as $x \to 0$, we may choose n_0 such that $\ell(Q_n) \leq \eta/2$ for $n \geq n_0$. Then

$$Q_{n+1} = Q_{n_0}\prod_{k=n_0}^{n}(m-\ell(Q_k)) \geq Q_{n_0}\left(m-\eta/2\right)^{n-n_0+1}$$

which is at least $(m-\eta)^{n+1}$ for n sufficiently large. This completes the proof. □

We are now ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1 (iii). We start with (a). Fix $\varepsilon \in (0,1)$. Take $h_n = (1+\varepsilon)\frac{\log(n)}{\log(1/m)}$. By Lemma 4.4 (ii), $nQ_{h_n} \leq n^{-\varepsilon} \to 0$. Now take $h_n = (1-\varepsilon)\frac{\log(n)}{\log(1/m)}$. Fix $\eta \in (0,m)$ such that

$$1-(1-\varepsilon)\frac{\log(m-\eta)}{\log m} \geq \frac{\varepsilon}{2}.$$

Then by Lemma 4.4 (ii), for n sufficiently large we have $nQ_{h_n} \geq n^{\varepsilon/2} \to \infty$. By applying Proposition 4.3, the claim of (a) follows.

Now we turn to (b). Let $u_n > 0$ be such that $Q_n = u_n m^n$. By [17, Theorem 2], u_n converges to a positive constant if $\sum_{n\geq 1}(n\log n)\mu_n < \infty$, and converges to 0 otherwise.

If $\sum_{n\geq 1}(n\log n)\mu_n < \infty$, it follows that $Q_n \sim cm^n$ as $n \to \infty$ for some $c > 0$, which by Proposition 4.3 implies that for every sequence $\lambda_n \to \infty$ we have

$$\mathbb{P}\left(\left|\mathsf{H}(\mathcal{T}_n)-\frac{\log(n)}{\log(1/m)}\right| \geq \lambda_n\right) \xrightarrow[n\to\infty]{} 0,$$

implying that the sequence $\big(\mathsf{H}(\mathcal{T}_n)-\log(n)/\log(1/m)\big)_{n\geq 1}$ is tight.

Now assume that $\sum_{n\geq 1}(n\log n)\mu_n = \infty$, so that $u_n \to 0$. We build a sequence $r_n \to \infty$ such that

$$\mathbb{P}\left(\mathsf{H}(\mathcal{T}_n) \leq \frac{\log(n)}{\log(1/m)}-r_n\right) \xrightarrow[n\to\infty]{} 1, \tag{4.4}$$

which will imply that the sequence $\big(\mathsf{H}(\mathcal{T}_n)-\log(n)/\log(1/m)\big)_{n\geq 1}$ is not tight. To this end, let $(a_p)_{p\geq 1}$ be an increasing sequence of integers such that for every $p \geq 1$, for every $n \geq a_p$ we have $c_n \leq 1/p$. Then, let $(p_n)_{n\geq 1}$ be the weakly increasing sequence of integers such that for every $n \geq 1$ we have

$$a_{p_n} \leq \frac{1}{2}\frac{\log(n)}{\log(1/m)} < a_{p_n+1}.$$

Observe that $p_n \to \infty$. Then set

$$r_n = \frac{1}{2\log(1/m)\min\big(\log(n), \log(p_n)\big)} \quad \text{and} \quad h_n = \frac{\log(n)}{\log(1/m)} - r_n.$$

Observe that $r_n \to \infty$ and that $h_n \geq \frac{1}{2}\frac{\log(n)}{\log(1/m)} \geq a_{p_n}$. Thus $u_{h_n} \leq 1/p_n$. As a consequence,

$$nQ_{h_n} \leq nu_{h_n}m^{h_n} \leq n\cdot\frac{1}{p_n}\cdot e^{-\log(n)+\log(1/m)r_n} \leq n\cdot\frac{1}{p_n}\cdot\frac{\sqrt{p_n}}{n} = \frac{1}{\sqrt{p_n}} \xrightarrow[n\to\infty]{} 0.$$

The convergence (4.4) then follows from Proposition 4.3 (ii) and this completes the proof. □

When $\sum_{n\geq 1}(n\log n)\mu_n = \infty$, in some particular cases it is possible to find the second order term in the magnitude of $\mathsf{H}(\mathcal{T}_n)$ by analysing the asymptotic behavior of $u_n = Q_n/m^n$ as $n \to \infty$, as seen in the following result.

Proposition 4.5. *Assume that $L(n) \sim c/\log(n)^{1+\beta}$ with $\beta \in (0,1]$.*

(i) For $\beta = 1$, for every $\varepsilon > 0$ we have

$$\mathbb{P}\left(\left|\mathsf{H}(\mathcal{T}_n) - \frac{\log(n)}{\log(1/m)} + \frac{c}{m\log(1/m)^2}\log\log n\right| \geq \varepsilon\log\log n\right) \xrightarrow[n\to\infty]{} 0.$$

(ii) For $\beta \in (0,1)$, for every $\varepsilon > 0$ we have

$$\mathbb{P}\left(\left|\mathsf{H}(\mathcal{T}_n) - \frac{\log(n)}{\log(1/m)} + \frac{c}{(1-\beta)\beta m\log(1/m)^{1+\beta}}\log(n)^{1-\beta}\right| \geq \varepsilon\log(n)^{1-\beta}\right) \xrightarrow[n\to\infty]{} 0.$$

Proof. Assume that $L(n) \sim c/\log(n)^{1+\beta}$ with $\beta \in (0,1]$. By Example 2.3 (i), and using the fact $\ell(x) \sim \ell^\star(1/x)$ as $x \to 0$ (this was seen in the proof of Lemma 4.4), the recurrence relation $Q_{n+1} = Q_n(m - \ell(Q_n))$ can be rewritten as

$$u_{n+1} = u_n\left(1 - \frac{c/(\beta m)}{\big(\log(1/u_n) + n\log(1/m)\big)^\beta}(1+\varepsilon_n)\right) \tag{4.5}$$

where ε_n is a sequence going to 0. Now we claim that $\log(1/u_n) = o(n)$. Indeed, by Lemma 4.4 for every $\eta \in (0,1)$ we have $\limsup(\log(1/u_n)/n) \leq \log(m/(m-\eta))$ and by taking $\eta \to 0$ we get our claim. Thus the recurrence relation (4.5) can be rewritten as

$$u_{n+1} = u_n\left(1 - \frac{c/(\beta m)}{\big(n\log(1/m)\big)^\beta}(1+\varepsilon'_n)\right) \tag{4.6}$$

where ε'_n is a sequence going to 0.

Now assume that $\beta = 1$. The recurrence relation (4.6) readily implies that $\log(u_n) \sim -\frac{c}{m\log(1/m)}\log(n)$. Set

$$h_n^\pm = \frac{\log(n)}{\log(1/m)} - \frac{c}{m\log(1/m)^2}\log\log n \pm \varepsilon\log\log n$$

and observe that $\log(u_{h_n^\pm}) \sim -\frac{c}{m\log(1/m)} \log\log n$ as $n \to \infty$. As a consequence,

$$\log\left(nQ_{h_n^\pm}\right) = \log(u_{h_n^\pm}) + \frac{c}{m\log(1/m)} \log\log n - (\pm\varepsilon \log(1/m)\log\log n),$$

which is asymptotic to $\mp\varepsilon\log(1/m)\log\log n$ as $n \to \infty$. The conclusion then follows from Proposition 4.3.

When $\beta \in (0,1)$, the recurrence relation (4.6) now implies that we have $\log(u_n) \sim -\frac{c}{(1-\beta)\beta m \log(1/m)^\beta} n^{1-\beta}$ and the desired result follows as in the case $\beta = 1$. □

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