i) Recurrence of $\mathcal{W}$

Let $S_n = Z_1 + \dots + Z_n$ be any random walk on $\mathbb{Z}$. Set $N_0 = \sum_{n\geq 0} \mathbb{1}_{\{S_n = 0\}} \geq 1$.

Thm Set $\varphi(t) = \mathbb{E}[e^{itZ_1}]$ for $t \in \mathbb{R}$.

(i) Either $\mathbb{P}(N_0 = \infty) = 1$ (we say that (Z_n) is recurrent) or $\mathbb{P}(N_0 < \infty) = 1$ ((Z_n) transient)

(ii) (Z_n) is transient $\iff \lim_{r \uparrow 1} \int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1 - r\phi(t)}\right) dt < \infty$

Proof: (i) Denote by $T_0 = \inf\{n \geq 1 ;\ S_n = 0\}$

By the Markov property, $\mathbb{P}(N_0 \geq k) = \mathbb{P}(T_0 < \infty)^k$ for every $k \geq 0$.

In particular, if $\mathbb{P}(T_0 < \infty) = 1$, then $\mathbb{P}(N_0 = \infty) = 1$. If $\mathbb{P}(T_0 < \infty) < 1$, write:

$\mathbb{E}[N_0] = \sum_{k\geq 0} \mathbb{P}(T_0 < \infty)^k = \frac{1}{1 - \mathbb{P}(T_0 < \infty)} < \infty$. Hence $N_0 < \infty$ a.s, i.e $\mathbb{P}(N_0 < \infty) = 1$.

(ii) The proof of (i) shows that

$$(Z_n) \text{ is transient} \iff \sum_{n\geq 0} \mathbb{P}(S_n = 0) = \mathbb{E}[N_0] < \infty.$$

We already saw that $\mathbb{P}(S_n = 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)^n dt$

(cf the proof of the Local Limit Theorem)

Hence
$$\sum_{n\geq 0} r^n \mathbb{P}(S_n = 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n\geq 0} (r\varphi(t))^n dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - r\varphi(t)} dt$$

Hence $\mathbb{E}[N_0] = \frac{1}{2\pi} \lim_{r\uparrow 1} \int_{-\pi}^{\pi} \frac{1}{1 - r\varphi(t)} dt.$ ☒

Q Is it true that

$$(Z_n) \text{ is transient} \iff \int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1 - \phi(t)}\right) dt < \infty \ ?$$

The direction $\boxed{\Rightarrow}$ is true: since $\mathrm{Re}\left(\frac{1}{1 - r\phi(t)}\right) \geq 0$,

by Fatou's lemma, gi

$$\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1 - \phi(t)}\right) dt \leq \lim_{r\uparrow 1} \int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1 - r\phi(t)}\right) dt$$

and the conclusion follows by the theorem.

The other direction is true (Chung-Fuchs Theorem), but is much more difficult.

Application: if $\mathbb{E}[S_1^2] < \infty$, $\mathbb{E}[S_1] = 0$, S_1 aperiodic, then (S_n) is recurrent.

Indeed, we know that $\frac{1}{1-\phi(t)}$ $|\phi(t)| < 1$ for $\in [-\pi, \pi] \setminus \{0\}$ (since Z_1 is aperiodic)

In addition $1 - \phi(t) = \frac{\sigma^2}{2} t^2 + o(t^2)$

Hence $\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{1}{1-\phi(t)}\right) dt = \infty$.

Hence (S_n) is recurrent. ☒

<u>NB</u> Actually $\mathbb{E}[|S_1|] < \infty$, $\mathbb{E}[S_1] = 0$, S_1 aperiodic

$\Rightarrow (S_n)$ recurrent.

<u>Corollary</u> $(W_n)_{n \geq 0}$ is recurrent.

<u>For more on random walks</u>:

<u>Principles of random walk</u>, Spitzer

<u>Probability Theory and examples</u> (chap 4), Durrett.

3). The space of continuous functions on $[0,1]$

If E is a metric space, let $\mathcal{B}_E$ be the Borel σ-field on E and $\mathcal{M}_1(E)$ be the set of probability measures on $(E, \mathcal{B}_E)$.

From now on, we work with

$$E = \mathcal{C}([0,1], \mathbb{R}),$$

the space of continuous functions from $[0,1] \to \mathbb{R}$ equiped with the uniform norm

$$(\text{if } f, g \in E, \quad \underbrace{d(f,g)}_{\|f-g\|_\infty =} = \sup_{0 \le s \le 1} |f(s) - g(s)|.)$$

For proofs and additional details, see

Convergence of probability measures by Billingsley

Proposition $(\mathcal{C}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ is complete and separable

(meaning that there exists a countable dense subset).

(E is polish)

For $0 \le t \le 1$, set $\pi_t : \begin{array}{l} E \longrightarrow \mathbb{R} \\ f \longmapsto f(t) \end{array}$

and denote by $\mathcal{F}_{cyl} := \sigma(\pi_t ; 0 \le t \le 1)$ the smallest σ-field on E for which π_t is measurable for every $0 \le t \le 1$.

Proposition. We have $\mathcal{B}_E = \mathcal{F}_{cyl}$

Proof : (⊂) $\pi_t : E \to \mathbb{R}$ is continuous, hence

$\pi_t : (E, \mathcal{B}_E) \to \mathbb{R}$ is measurable.

Hence $\mathcal{F}_{cyc} \subset \mathcal{B}_E$

(⊃) Take $f \in E$, $r > 0$. We show that $B_E(f; r) \in \mathcal{F}_{cyc}$.

closed ball in E, of radius r, centered in f.

$$B_E(f; r) = \{ g \in E ; \|f - g\| \le r \}$$

$$= \{ g \in E ; \sup_{0 \le s \le 1} |f(s) - g(s)| \le r \}$$

$= \{ g \in E ; \sup_{\substack{0 \le s \le 1 \\ s \in \mathbb{Q}}} |f(s) - g(s)| \le r \}$

$= \bigcap_{s \in [0,1] \cap \mathbb{Q}} \pi_s^{-1} (B_{\mathbb{R}}(f(s); r))$

$\in \mathcal{F}_{cyc}$.

Since E is separable, every open set of E is a countable union of closed balls, so that $O \in \mathcal{F}_{cyc}$ if O is open.

Hence $\mathcal{B}_E \subset \mathcal{F}_{cyc}$ ∎

Remark Let $\mathcal{C}_{cyl}$ be the class of sets of the form

$\{ f \in E ; f(t_1) \in A_1, \dots, f(t_k) \in A_k \}$

with $0 \le t_1 \le \dots \le t_k \le 1$, $A_1, \dots, A_k \in \mathcal{B}(\mathbb{R})$.

[Cylinder sets]

then $\mathcal{C}_{cyl}$ is closed under finite intersections and $\sigma(\mathcal{C}_{cyl}) = \mathcal{F}_{cyl}$.

As a consequence, if $\mu, \nu \in \mathcal{M}_1(E)$ are such that

$\mu(A) = \nu(A)$ for every $A \in \mathcal{C}_{cyl}$, then $\mu = \nu$

Idea: The cylinder σ-field contains all the information you can get by evaluating function at a fixed $\boxed{\text{countable}}$ number of times.

What about convergence of probability measures of $\mathcal{M}_1(E)$?

Recall that if $\mu_n, \mu \in \mathcal{M}_1(E)$, then $\mu_n \xrightarrow{\text{weakly}} \mu$

if for every continuous bounded function $F : E \to \mathbb{R}$,

we have $\int_E F(f) \mu_n(df) \xrightarrow[n \to \infty]{} \int_E F(f) \mu(df)$.

Thm [Prokorhov] Assume that E is polish. Let $(\mu_n) \in \mathcal{M}_1(E)$. Then (i) $\Leftrightarrow$ (ii)

(i) there exists a subsequence $n_k \to \infty$ and $\mu \in \mathcal{M}_1(E)$ s.t. $\mu_{n_k} \xrightarrow{\text{weakly}} \mu$

(ii) (μ_n) is tight, i.e. $\forall \varepsilon > 0$, $\exists K_\varepsilon \subset E$ compact such that

$$\limsup_{n \to \infty} \mu_n(E \setminus K_\varepsilon) \le \varepsilon .$$