Remark 1.8: It is possible to prove the following stronger result (useful in particular if \( k \) is very large):

\[
\sup_{x \in \mathbb{Z}} \max \left( 1, \frac{(k-x)^2}{\sqrt{n}} \right) \sim \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{(k-x)^2}{\sqrt{n}} \right) \quad n \to \infty
\]

See *Principles of Random Walk*, Chapter 7, by Spitzer.

Remark 2: When \( n = \infty \), it is possible to extend the Local Limit Theorem when the law of \( Z_n \) belongs to the domain of attraction of a stable law of index \( 0 < \alpha < 2 \). If \( \mathbb{P}(Z_1 < -1) = 0 \), this is equivalent to the fact that \( \mathbb{P}(Z_1 > k) \sim \frac{L(k)}{k^d} \) where \( L \) satisfies

\[
\frac{L(\alpha n)}{L(n)} \to 1 \quad n \to \infty
\]

for every \( d > 0 \).

6) Applications of the Local Limit Theorem

Recall that \( W_n = X_1 + \ldots + X_n \) with \( \mathbb{P}(X_i = 1) = \mu(i), i > 1 \) and that \( \mathbb{E}X_i = \sum_{i > 1} \mu(i) = -1 \) \( i > 1 \), \( i > 0 \)

Assume that \( \mu \) is critical and has finite variance \( \sigma^2 \). Then \( \mathbb{E}X_1^2 = 0 \), \( \mathbb{E}X_1^3 = \sum_{i > 1} i^2 \mu(i) = \sum_{i > 0} (i-1)^2 \mu(i) = \sum_{i > 0} i^2 \mu(i) - 1 = \sigma^2 \).
**Definition:** We say that $\mu$ is aperiodic if the maximal integer $h \geq 1$ such that $\sum_{i \geq 1} \mu(i) > 0 \forall c \in \mathbb{Z}$ is $h=1$. Or, equivalently, that $\gcd(i, 1) > 1; \mu(i) > 0 \forall c = 1.$

**Lemma:** Assume that $\mu$ is aperiodic and $\mu(0) > 0$.

Then $\exists N, s.t. n \geq N \Rightarrow P_{\mu}(1 \mid = n) > 0$

**Proof:** by aperiodicity, we can find $i$ and $j$ s.t.

$\mu(i) > 0, \mu(j) > 0$ and $\gcd(i, j) = 1$.

Take $N = i, j$.

Fix $n \geq N+1$, let $0 \leq b \leq i-1$ be the integer such that $b = n \mod i$ (possible since $\gcd(i, j) = 1$)

Hence there exists $a \in \mathbb{Z}$ such that $n-1 - bj = ai$.

But $bj < c \leq n-1$. Hence $a > 0$.

Thus $n-1 = ai + bj$ with $a > 0, b > 0$.

Hence with positive probability a CW tree has a vertex with $i$ children, $b$ vertices with $j$ children and all the other vertices $0$ children. But such a tree has $n$ vertices in total. Hence $P_{\mu}(1 \mid = n) > 0$.

**Remark:** If $\mu$ aperiodic, then $X_i$ is aperiodic (in the same defined in Lecture 5).

(Indeed, if $\exists$ support $x_i < c + n \mathbb{Z}$, then $\exists i > 1; \mu(i) > 0 \not\exists c \mid 1 + n \mathbb{Z}$ and $\mu(i) > 0, \mu(j) > 0 \Rightarrow h \lor j$)

**Application of the Local Limit Theorem to the study of the size of CW trees**

Assume that $\mu$ is critical, aperiodic, with finite variance.

We have seen that $P_{\mu}(1 \mid = n) = \frac{1}{n} P(W_n = -1) \sim \frac{1}{\sqrt{n \pi}} e^{-\frac{1}{2} \left(\frac{1}{n} \right)^2} + \frac{\varepsilon(k, n)}{\sqrt{n}}$.

By the local limit theorem, $P(W_n = -1) \sim \frac{1}{\sqrt{2 \pi n}} e^{-\frac{1}{2} \left(\frac{1}{n} \right)^2} + \frac{\varepsilon(k, n)}{\sqrt{n}}$.

With $\sup_{k \in \mathbb{Z}} \varepsilon(k, n) \to 0$. Hence $P(W_n = -1) \sim \frac{1}{\sqrt{2 \pi n}} e^{-\frac{1}{2} \left(\frac{1}{n} \right)^2} + C \sqrt{n}.$
Here \( \beta_p(111^n) \sim N \left( \frac{1}{\sqrt{2\pi} \sigma^2} \right) \frac{1}{n^{3/2}} \)

(in particular, note that \( E[111^n] = \infty \), which is consistent with the fact that \( E[111^n] = E[1 \geq 2n^2] = \sum_{n=0}^{\infty} E[1 \geq 2n^2] = \sum_{n=0}^{\infty} 1 = \infty \)

\[ \sum_{n} 1 = \infty \]

Application to counting dissections

Recall the construction of the dual tree \( T_0 \) of a dissection \( D_n \):

If \( D_n \) is a uniform dissection of \( P_n \), we have seen that the law of \( T(D_n) \) is \( \beta_p(1 \mapsto \lambda(T)) = n-1 \)

where \( \mu(0) = \frac{1-2c}{2c} \), \( \mu(1) = 0 \), \( \mu(i) = c^{i-1} \) for \( i > 2 \).

We choose \( c \) so that \( \mu \) is critical: \( c = 1 - \frac{\sqrt{2}}{2} \)

then \( \mu(0) = 2 - \sqrt{2} \), \( \mu(1) = 0 \), \( \mu(i) = \left( \frac{2 - \sqrt{2}}{2} \right)^{i-1} \) for \( i > 2 \).

Hence:

\[
\frac{1}{|D_n|} = \beta_p(T = \quad \lambda \mapsto \lambda(n) = n-1) = \frac{\mu(n-1) \cdot \mu(0)}{\bar{\beta}(\lambda(T) = n-1)}
\]

Hence \( |D_n| = \frac{(2-\sqrt{2})^3}{(3-2\sqrt{2})^n} \frac{1}{\mu(0)} \beta_p(\lambda(T) = n-1) \).

How can we estimate \( \beta_p(\lambda(T) = n-1) \)?

We use a construction due to Rzgolo (2011) which maps a \( T \) to \( W \) tree with \( n \) leaves to a \( W \) tree with \( n \) vertices:

Start with a tree \( T \) with \( n \) leaves. Then, iteratively, "collapse" the edges of \( T \) that lead to the left-most leaf.
Rizzolo showed that if $\tilde{T}$ is a $GW_{\mu}$ tree, then $\tilde{T}$ is a $GW_{\mu_0}$ tree, and:

$\tilde{\nu}$ critical with variance $\sigma^2$.

Hence

$$P_{\tilde{\nu}}(X(T) = n-1) = \frac{1}{\tilde{\nu}} \frac{1}{\sqrt{2\pi n \sigma^2}} \frac{1}{n^{3/2}}.$$

We conclude by putting the pieces together that

$$\frac{1}{P_{\tilde{\nu}}(X(T) = n-1)} \cdot \sqrt{\frac{8 (3+2\sqrt{2})}{11\pi} n^{3/2}} \sim \frac{3}{2} \sqrt{\frac{8 (3+2\sqrt{2})}{11\pi}} n^{-3/2}$$

(2) Number of children of the root.

We know that $P_{\mu}(R_X = i) = \mu(i)$. But what about $P_{\mu}(R_X = i \mid X(T) = n)$ as $n \to \infty$?

The coding by the Lukasiewicz path gives that

$$P_{\mu}(R_X = i \mid X(T) = n) = \frac{P(X_1 = i-1 \mid Z_1 = n)}{P(Z_1 = n)} \frac{1}{\mu(i)} \cdot P(Z_1 = n-1)$$

$$\to i \mu(i) \frac{n}{n-1} \frac{1}{P(W_{n-1} = i)} \frac{P(W_{n-1} = -i)}{P(W_{n-1} = -i)} = \frac{\mu(i)}{n \to \infty}$$ by the Local Limit Theorem.
Local limits of Galton–Watson trees.

Here \( \mu \) is an offspring distribution on \( \mathbb{N}_+ \). We assume that \( \mu \) is aperiodic.

1) Critical and finite variance case.

Assume that \( \sum \mu(i) = 1 \) and \( \sigma^2 = \sum i^2 \mu(i) - 1 \in (0, \infty) \).

Let \( T_n \) be a random tree with law \( \omega_{W_\mu}(\cdot | |T|=n) \).

Goal: \( T_n \) converges "locally" to an \( \infty \)-tree. \( T_\infty \)

Define \( T_\infty \) as the following \( \infty \)-tree:

- There are 3 types of vertices: normal and mutant.
- The number of children of normal vertices has law \( \mu \), and they are normal.
- The number of children of mutant vertices have law \( \overline{\mu} \) defined by \( \overline{\mu}(i) = i \mu(i) \), and all are normal, except one which is mutant and which is chosen uniformly among the children.

For every \( t \in T_n \), set \( Z_k(t) = \sum_{i \in \mathbb{N}_+} i, |u| = k, 1 \leq k \leq |T_n| \).

Prop: For every finite tree \( t_0 \),

\[ P_k(\sum_{k} T_{\infty} = t_0) = Z_k(t_0) P_{\mu}(\sum_{k} T_k = \infty) \]

Proof: exercise.
Assume that $p$ is critical, periodic, has finite variance.

For every finite tree $Z_0$ and $k \geq 0$,

$$\Rightarrow \quad \mathbb{P}(\{ T_n \}_{k} = Z_0) \quad n \to \infty \quad \mathbb{P}(\{ T_{\infty} \}_{k} = Z_0)$$

We say that $T_n$ converges locally in distribution to $T_\infty$.

Remarks:

- The local convergence may be defined by a metric.
- If a distinguished point is not given, one can choose it uniformly at random, in which case one speaks of local weak convergence.

Proof: By the proposition, it is enough to check that for every tree $Z_0$ of height $k$, we have

$$\mathbb{P}(\{ T_n \}_{k} = Z_0) \rightarrow \mathbb{P}_{\infty}(\{ T_{\infty} \}_{k} = Z_0) \mathbb{P}_{\infty}(1 \leq |n-k|)$$

But

$$\mathbb{P}(\{ T_n \}_{k} = Z_0) = \mathbb{P}_{\infty}(\{ T_{\infty} \}_{k} = Z_0) \times \mathbb{P}_{\infty}(1 \leq |n-k|)$$

This completes the proof.

Remark: the theorem is true even if $\sigma^2 = \infty$, indeed it is known that $W_n, k \in \mathbb{Z}$.

$$\Rightarrow \quad \mathbb{P}(W_{n-m} = -k) \quad n \to \infty \quad \mathbb{P}(W_{n} = -1)$$