Lecture 4

• Recall that $\chi : \overline{S}_n^{(k)} \times \mathbb{Z}/n\mathbb{Z} \to \{(\underline{x}, i) ; \underline{x} \in S_n^{(k)}, i \in I_{\underline{x}}\}$
$$(\underline{x}, i) \mapsto (\underline{x}^{(i)}, -i)$$
is a bijection ($\chi^{-1}(\underline{x}, i) = (\underline{x}^{(i)}, -i)$).

• If $\varphi : A \to B$ is a bijection between two finite sets A and B, we have

$$(*) \quad \sum_{a \in A} G(a) = \sum_{b \in B} G(\varphi^{-1}(b)) \text{ for every } G : A \to \mathbb{R}.$$

Lemma: Let $F : S_n^{(k)} \to \mathbb{R}$ be a function invariant by cyclic shifts (i.e. $F(\underline{x}) = F(\underline{x}^{(i)})$ for every $x \in S_n^{(k)}$ and $i \in \mathbb{Z}/n\mathbb{Z}$). Then:

$$\sum_{\underline{x} \in \overline{S}_n^{(k)}} F(\underline{x}) = \frac{k}{n} \sum_{\underline{x} \in S_n^{(k)}} F(\underline{x})$$

Proof: Write $\displaystyle\sum_{\underline{x} \in \overline{S}_n^{(k)}} F(\underline{x}) = \frac{1}{n} \sum_{\substack{\underline{x} \in \overline{S}_n^{(k)} \\ i \in \mathbb{Z}/n\mathbb{Z}}} F(\underline{x}) = \frac{1}{n} \sum_{\substack{\underline{x} \in S_n^{(k)} \\ i \in I_{\underline{x}}}} F(\underline{x}^{(i)})$ by $(*)$

$$= \frac{1}{n} \sum_{\substack{\underline{x} \in S_n^{(k)} \\ i \in I_{\underline{x}}}} F(\underline{x}) \quad (\text{since } F(\underline{x}) = F(\underline{x}^{(i)}))$$

$$= \frac{k}{n} \sum_{\underline{x} \in \overline{S}_n^{(k)}} F(\underline{x}) \quad (\text{since } |I_{\underline{x}}| = k \text{ by the cyclic lemma})$$

∎

We now return to the proof of Lagrange's inversion theorem: recall that $\phi(z) = \sum_{n \geq 0} a_n z^n$ and that $w(z) = \sum_{t \in \mathbb{T}_f} z^{|t|} \cdot \prod_{u \in t} a_{k_u}$.

We show that $\boxed{[z^n]\, w(z)^k = \frac{k}{n} [z^{n-k}]\, \phi(z)^n}$

• First, since $\phi(z)^n = \displaystyle\sum_{i_1, \dots, i_n \geq 0} a_{i_1} \cdots a_{i_n} z^{i_1 + \dots + i_n}$, we have

$$\frac{k}{n} [z^{n-k}]\, \phi(z)^n = \frac{k}{n} \sum_{\substack{i_1 + \dots + i_n = n-k \\ i_1, \dots, i_n \geq 0}} a_{i_1} \cdots a_{i_n}$$

$$= \frac{k}{n} \sum_{\substack{i_1 + \dots + i_n = -k \\ i_1, \dots, i_n \geq -1}} a_{i_1+1} \cdots a_{i_n+1}$$

$$= \frac{k}{n} \sum_{\underline{x} \in S_n^{(k)}} \prod_{j=1}^{n} a_{x_j+1}$$

• Next, since $w(z)^k = \sum_{t_1,\dots,t_k \in \mathcal{T}_y} z^{|t_1|+\dots+|t_k|} \prod_{u \in t_1,\dots,t_k} a_{k_u}$,

we have $$[z^n]\, w(z)^k = \sum_{f \in \mathbb{F}_n^{(k)}} \prod_{u \in f} a_{k_u}$$

But $$\Phi : \mathbb{F}_n^{(k)} \longrightarrow \bar{S}_n^{(k)}$$
$$f = (t_1,\dots,t_k) \longmapsto (k_{u_0}(t_1)-1, \dots)$$
is a bijection.

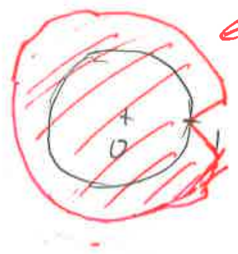
Hence, by $(*)$, $$[z^n]\, w(z)^k = \sum_{x \in \bar{S}_n^{(k)}} \prod_{j=1}^{n} a_{x_j+1}.$$

But $$F : S_n^{(k)} \longrightarrow \mathbb{R}, \quad x \longmapsto \prod_{j=1}^{n} a_{x_j+1}$$ is invariant by cyclic shifts.

Hence $$[z^n]\, w(z)^k = \frac{k}{n} \sum_{x \in S_n^{(k)}} \prod_{j=1}^{n} a_{x_j+1}$$

which is $\frac{k}{n}[z^{n-k}]\,\phi(z)^n$ by the first step! ■

3) Transfer Lemmas



← Δ domain

Prop Assume that f is analytic in Δ and that $f(z) \underset{z\to 1}{\sim} \frac{1}{(1-z)^\alpha}$ with $\alpha \neq 0,-1,-2,\dots$

Then $$[z^n]\, f(z) \underset{n\to\infty}{\sim} \frac{n^{\alpha-1}}{\Gamma(\alpha)}$$

See Analytic Combinatorics by Flajolet & Sedgewick.

NB If we have $f(z) \underset{z\to r}{\sim} \frac{C}{(r-z)^\alpha}$, we apply the proposition with $g(z) := f(zr)$, and if g is analytic in a Δ domain, we get that $[z^n]\, g(z) \underset{n\to\infty}{\sim} \frac{C}{r^\alpha} \cdot \frac{n^{\alpha-1}}{\Gamma(\alpha)}$, so that $[z^n]\, f(z) \underset{n\to\infty}{\sim} \frac{C}{r^\alpha} \cdot \frac{r^n\, n^{\alpha-1}}{\Gamma(\alpha)}$

Application: we now that $E(z) = \sum_{n \geq 2} |D_n|\, z^{n-1} = (1+z+\sqrt{1-6z+z^2})/4$

$\Rightarrow |D_n| \underset{n\to\infty}{\sim} \frac{1}{4\sqrt{\pi}} \sqrt{99\sqrt{2}-140}\, (3+2\sqrt{2})^n \cdot n^{-3/2}$

IV Galton-Watson trees and their coding by random walks.

In this section, μ will be a probability measure on $\mathbb{Z}_+$ such that $\boxed{\mu(0)+\mu(1)<1}$ and $\boxed{m := \sum_{k\geq 0} k\mu(k) \leq 1}$

Set $\sigma^2 = \sum_{k\geq 0} k^2 \mu_k - m^2 \in (0,\infty]$.

1) Definition

Recall that $\Pi_f = \{$ finite plane rooted trees $\}$.

Proposition / Definition.
For every $\underline{t} \in \Pi_f$, set $\mathbb{P}_\mu(\underline{t}) = \prod_{u\in \underline{t}} \mu(k_u)$.
Then $\mathbb{P}_\mu$ is a probability measure on Π_f

Proof: Let $(X_u ; u \in \mathcal{U})$ be an iid collection of random variables with distribution μ.

Define $\Upsilon := \{ u_1 \dots u_n \in \mathcal{U} ;\ u_i \leq X_{u_1 \dots u_{i-1}} \ \forall 1 \leq i \leq n \}$

Then Υ is a plane rooted, possibly infinite, tree which is random.

Set $Z_n = |\{ u \in \Upsilon ;\ |u| = n \}|$

Then $(Z_n)_{n\geq 0}$ is a GW_μ process is extinct almost surely since $m \leq 1$.

Hence $\mathbb{P}(\Upsilon \in \Pi_f) = 1$.

But
$$\mathbb{P}(\Upsilon = \underline{t}) = \mathbb{P}(X_u = k_u \quad \forall u \in \underline{t}) = \prod_{u\in \underline{t}} \mu(k_u)$$
for every $\underline{t} \in \Pi_f$.

∎

2) Examples.

- Take $\mu(0) = \mu(2) = \frac{1}{2}$. Then a uniform binary (plane rooted) tree with n vertices has law $\mathbb{P}_\mu(\,\cdot \mid |T| = n)$.

Proof Let $\underline{t}$ be a binary trees with n vertices (n odd) it is enough to check that $\mathbb{P}_\mu(\underline{t})$ only depends on n.

But $\mathbb{P}_\mu(\underline{t}) = \prod_{u\in\underline{t}} \mu_{k_u} = \prod_{u\in\underline{t}} \frac{1}{2} = \frac{1}{2^n}$. $\boxtimes$

2) A uniform tree with n vertices has law $\mathbb{P}_\mu(\,\cdot \mid |\mathcal{T}|=n)$ with $\mu(i)=\frac{1}{2^{i+1}}$ for $i\geqslant 0$.

__Proof:__ Take a tree $\underline{t}$ with n vertices. It is enough to check that $\mathbb{P}_\mu(\underline{t})$ depends only on n.

$$\mathbb{P}_\mu(\underline{t}) = \prod_{u\in\underline{t}} \mu_{k_u} = \prod_{u\in\underline{t}} \frac{1}{2^{k_u+1}} = \frac{1}{2^{\sum_{u\in\underline{t}}(k_u+1)}} = \frac{1}{2^{2n-1}} \quad \boxtimes$$

3) A uniform rooted non-ordered (non plane) tree with n vertices has the law of $\text{Shape}(\mathcal{T}_n)$ where $\mathcal{T}_n$ is a $\mathbb{P}_\mu(\,\cdot \mid |\mathcal{T}|=n)$ tree with $\mu(i)=\frac{e^{-1}}{i!}$ and $\text{Shape}(\mathcal{T}_n)$ is the non-ordered tree obtained from $\mathcal{T}_n$.

__Proof__ Take $\underline{t}^\circ$ a non-ordered rooted tree with n vertices then

$$\mathbb{P}_\mu(\text{Shape}(\mathcal{T}_n)=\underline{t}^\circ) = \mathbb{P}_\mu(\exists \text{ ordering } \underline{t} \text{ of } \underline{t}^\circ \text{ such that } \mathcal{T}_n=\underline{t})$$

$$= \sum_{\underline{t} \text{ ordering of } \underline{t}^\circ} \mathbb{P}_\mu(\mathcal{T}_n=\underline{t}) = \underbrace{\sum_{\underline{t} \text{ ordering of } \underline{t}^\circ}}_{\substack{\text{We have } \prod_{u\in\underline{t}^\circ} k_u! \\ \text{possible orderings}}} \underbrace{\prod_{u\in\underline{t}} \frac{e^{-1}}{k_u!}}_{\text{Same for any ordering}}$$

$$= e^{-n}$$

4) More generally, let $(\lambda(i))_{i\geqslant 0}$ be any sequence of $\mathbb{R}_+$. For $\underline{t}\in\mathbb{T}_f$, set $\omega(\underline{t}) = \prod_{u\in\underline{t}} \lambda(k_u)$ and

$$Z_n = \sum_{|t|=n} \omega(\underline{t})$$

If $Z_n>0$, define $\mathbb{P}_\lambda^{(n)}(\underline{t}) = \frac{1}{Z_n}\,\omega(\underline{t})$, which is a probability measure on $\mathbb{T}_n$. This is the model of simply generated trees