Exercise given at the end of Lecture 8.

Let μ be a critical and aperiodic offspring distribution on $\mathbb{Z}_+$, with finite and positive variance σ^2. Recall that:

* $W_n = X_1 + \cdots + X_n$ is the random walk such that $\mathbb{P}(X_1 = i) = \mu(i+1)$ for $i \geq -1$.
* $T_0 = 0$ and $T_k = \inf\{i > T_{k-1} ; W_i \geq W_{T_{k-1}}\}$ for $k \geq 1$ are the weak record times of W.
* $H_n = |\{0 \leq k \leq n-1 ; W_k = \min_{k \leq j \leq n} W_j\}|$ for $n \geq 0$ is the associated height process.

(i) Calculate $\mathbb{E}\left[W_{T_1}\right]$.

(ii) Set $I_n = \min_{0 \leq i \leq n} W_i$. Show that

$$\frac{H_n}{W_n - I_n} \xrightarrow[n\to\infty]{(\mathbb{P})} \frac{2}{\sigma^2},$$

where $(\mathbb{P})$ means that the convergence holds in probability.