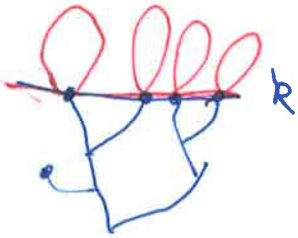
Solution to the exercise

1) As in the lecture, it is enough to show that for every $k \geq 0$, if $\underline{t}$ has height k, then

$$\mathbb{P}([T_n]_k = \underline{t}) \longrightarrow Z_k(\underline{t})\, \mathbb{P}_\mu([T]_k = \underline{t})$$

number of individuals at height k in $\underline{t}$.

But $\mathbb{P}([T_n]_k = \underline{t}) = \dfrac{1}{\mathbb{P}_\mu(\lambda(T)=n)} \mathbb{P}_\mu([T]_k = \underline{t}$ and T has n leaves$)$



$$= \frac{1}{\mathbb{P}_\mu(\lambda(T)=n)} \mathbb{P}_{\mu, Z_k(\underline{t})}(F \text{ has } n-q \text{ leaves}) \cdot \mathbb{P}_\mu([T]_k = \underline{t})$$

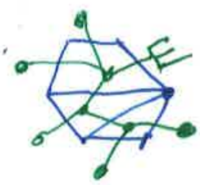
(where q is the number of leaves in $\underline{t}$ with height $\leq k-1$)

$$= \mathbb{P}_\mu([T]_k = \underline{t}) \cdot \frac{\mathbb{P}_{\tilde{\mu}, Z_k(\underline{t})}(|F| = n-q)}{\mathbb{P}_{\tilde{\mu}}(|T| = n)}$$

by Rizzolo's result (with $\tilde{\mu}$ critical with finite variance)

$$\xrightarrow[n\to\infty]{} Z_k(\underline{t}) \cdot \mathbb{P}_\mu([T]_k = \underline{t}) \text{ as in the lecture.}$$

2)



We know that the dual tree $\mathcal{T}_n$ of $\mathcal{D}_n$ is a $\mathbb{P}_\mu(\cdot \mid \lambda(T) = n-1)$ tree, with $\mu(0) = 2-\sqrt{2}$, $\mu(1) = 0$, $\mu(i) = \left(\frac{2-\sqrt{2}}{2}\right)^{i-1}$, $i \geq 2$ which is critical and has finite variance.

Hence $[\mathcal{T}_n]_k \xrightarrow{(d)} [\mathcal{T}_\infty]_k$ for every $k \geq 0$ by 1)

a) $\tilde{\Delta}_n$ = number of children of the root $+1$ in $\mathcal{T}_n$.
Hence $\mathbb{P}(\tilde{\Delta}_n = k) \longrightarrow (k-1)\mu(k-1)$ for $k \geq 3$, hence the result

b) $\mathcal{V}_n$ = length of the left-most path from the root in $\mathcal{T}_n - 1$.

Hence $\mathcal{V}_n \xrightarrow[n\to\infty]{(d)} \ell_\infty$ = length of the left-most path from the root in T_∞.

Write $\ell_\infty = \ell_1 + \ell_2$, where ℓ_1 is the smallest integer i such that $\underbrace{1 \cdots 1}_{i}$ is not on the spine of T_∞, and ℓ_2 is the length of the left-most path from the root in a GW_μ tree, and $\boxed{\ell_1 \perp\!\!\!\perp \ell_2}$

(*) It is clear that $\mathbb{P}(\ell_2 \geq k) = (1-\mu_0)^k$

(*) Also, $\mathbb{P}(\ell_1 \geq k) = \mathbb{P}($the spine goes through the left most child in T_∞ for the first k generations$)$

$$= \Big(\sum_{i \geq 1}^{\infty} \underbrace{i\mu(i)}_{i \text{ children}} \times \underbrace{\frac{1}{i}}_{\text{the left-most is selected}} \Big)^k$$

$$= (1-\mu_0)^k.$$

The conclusion follows.