Exercice given at the end of Lecture 5.

Recall μ is offspring distribution on $\mathbb{Z}_+$, that $W_n = X_1 + \cdots + X_n$ is a random walk on $\mathbb{Z}$ with $\mathbb{P}(X_1 = i) = \mu(i+1)$ for $i \geq -1$, and if F is a forest $u_1(F), \ldots, u_{|F|}(F)$ are its vertices ordered in lexicographical order.

Fix an integer $j \geq 1$ and let $F : \mathbb{Z}^n \to \mathbb{R}_+$ be a function invariant under cyclic shifts (meaning that $F(\mathbf{x}) = F(\mathbf{x}^{(i)})$ for every $\mathbf{x} \in \mathbb{Z}^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$), and let $\mathcal{F}$ be a forest of j independent GW_μ random trees. Show that:

$$\mathbb{E}\left[F\left(k_{u_0}(\mathcal{F}) - 1, k_{u_2}(\mathcal{F}) - 1, \ldots, k_{u_{n-1}}(\mathcal{F}) - 1\right) \mathbb{1}_{\{|\mathcal{F}|=n\}}\right] = \frac{j}{n} \mathbb{E}\left[F(X_1, \ldots, X_n) \mathbb{1}_{\{W_n = -j\}}\right].$$