Solution to the exercise

A forest $f$ with $k$ trees is a sequence $f = (e_1, \ldots, e_k)$ of $k$ forests.

Recall the bijection $\phi_n : T_n \rightarrow \overline{S}_n$ where $\overline{l} \mapsto (k_{u_0}(l), \ldots, k_{u_{11-1}}(l))$.

Denote by $F_n^{(k)}$ the set of all forests with $k$ trees and $n$ vertices.

Then set $\Phi : F_n^{(k)} \rightarrow \overline{S}_n^{(k)}$ where $\overline{e_1}, \ldots, \overline{e_k} \mapsto \phi_{e_1}(\overline{e_1}) \cdot \ldots \cdot \phi_{e_k}(\overline{e_k})$.

For example, $\overline{5} = \overline{5}$, $\overline{1} = \overline{1}$, $\overline{1} = \overline{1}$.

Then $\Phi(\overline{\overline{5}}) = \overline{5}, \overline{1}, -1, 0, 0, -1$.

And the associated Lukasiewicz path is

Then $\Phi$ is a bijection (similar proof).

Hence $|F_n^{(k)}| = |\overline{S}_n^{(k)}| = \binom{2n-k-1}{n-1} \times \frac{k}{n}$.