Exercise: Let $(Z_n)_{n\geq 0}$ be a Galton-Watson process with offspring distribution μ.

a) Calculate $\mathbb{E}[Z_n]$

Solution ①

Use the definition:

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}\Big[\sum_{j=1}^{Z_n} X_j^{(n)}\Big] = \sum_{k=0}^{\infty} \mathbb{E}\Big[\sum_{j=1}^{k} X_j^{(n)} \mathbb{1}_{Z_n=k}\Big]$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Z_n=k)\, \mathbb{E}\Big[\sum_{j=1}^{k} X_j^{(n)}\Big] \quad \text{since } Z_n \perp (X_1^{(n)},\dots,X_k^{(n)})$$

$$= \sum_{k=0}^{\infty} k\, \mathbb{P}(Z_n=k)\, m \qquad \text{where } m = \mathbb{E}[X_1^{(1)}] = \mathbb{E}[Z_1]$$

$$= \mathbb{E}[Z_n]\, m.$$

Hence, by induction, $\boxed{\mathbb{E}[Z_n] = m^n}$ for $n \geq 0$

Solution ② Use the fact that $\phi_n(s) = \mathbb{E}[s^{Z_n}]$

Hence $\phi_n'(s) = \mathbb{E}[Z_n \cdot s^{Z_n - 1}]$ for $s \in [0,1)$.

By taking the limit $s \uparrow 1$, By monotone convergence we have

$$\phi_n'(1) = \mathbb{E}[Z_n].$$

But $\phi_{n+1}(s) = \phi(\phi_n(s)) \Rightarrow \phi_{n+1}'(s) = \phi_n'(s)\cdot \phi'(\phi_n(s))$

Hence $(s=1)$ gives $\phi_{n+1}'(1) = \phi_n'(1)\,\phi'(1) = m\,\phi_n'(1)$.

By induction, we get $\phi_n'(1) = m^n$ for $n \geq 0$.

b) Calculate $\mathbb{E}[Z_n^2]$

First, if $\mathbb{E}[Z_n] = \infty$, then $\mathbb{E}[Z_n^2] = \infty$, hence we suppose $m < \infty$.

Solution: We use the idea of solution ② above:

We have $\phi_n''(s) = \mathbb{E}[Z_n \cdot (Z_n - 1)\, s^{Z_n - 2}]$ for $0 \leq s < 1$,

and by the same argument as above,

$$\phi_n''(1) = \mathbb{E}[Z_n(Z_n-1)] = \mathbb{E}[Z_n^2] - \mathbb{E}[Z_n] = \mathbb{E}[Z_n^2] - m^n.$$

It thus enough to find $\phi_n''(1)$.

We have $\phi_{n+1}(s) = \phi_n(\phi(s))$.

Hence $\phi_{n+1}'(s) = \phi'(s)\, \phi_n'(\phi(s))$

Hence $\phi_{n+1}''(s) = \phi''(s)\, \phi_n'(\phi(s)) + (\phi'(s))^2\, \phi_n''(\phi(s))$

$s=1$ gives $\phi_{n+1}''(1) = \phi''(1)\, \phi_n'(1) + (\phi'(1))^2\, \phi_n''(1)$.

But we know that $\phi_n'(1) = m^n$ and $\phi'(1) = m$.

then $\boxed{\phi_{n+1}''(1) = \phi''(1)\, m^n + m^2\, \phi_n''(1)}$

This is a recursion relation, which has the solution

$$\phi_n''(1) = \phi''(1)\, m^{n-1} \cdot \frac{1-m^n}{1-m} \quad \left(= \sigma^2 m^{n-1} (1 + m + \dots + m^{n-1}) \right)$$

Hence $\boxed{\mathbb{E}[Z_n^2] = \phi''(1)\, m^{n-1} \cdot \frac{1-m^n}{1-m} + m^n.}$

But $\phi''(1) = \mathbb{E}[Z_1^2] - m$, so it is also possible to write

$$\boxed{\mathbb{E}[Z_n^2] = (\mathbb{E}[Z_1^2] - m) \cdot m^{n-1} \cdot \frac{1-m^n}{1-m} + m^n}$$