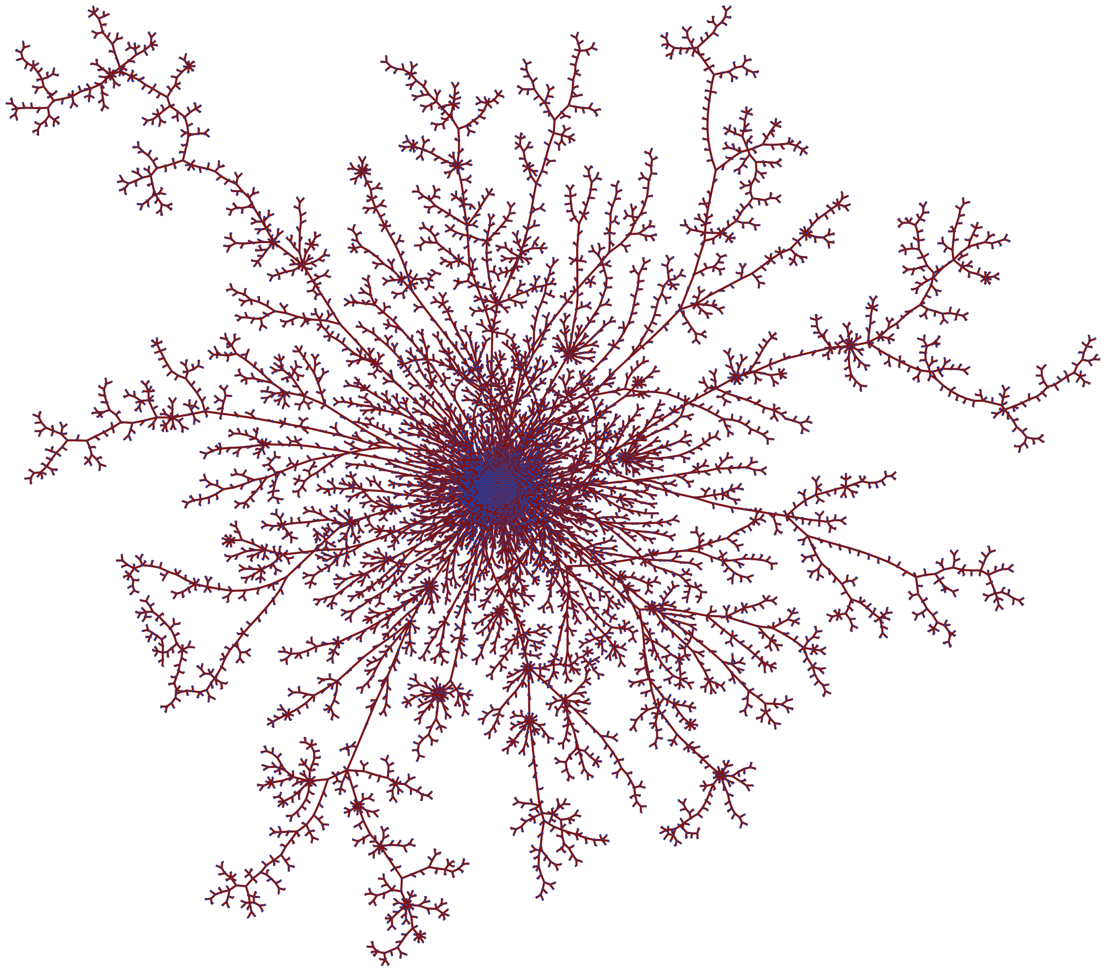
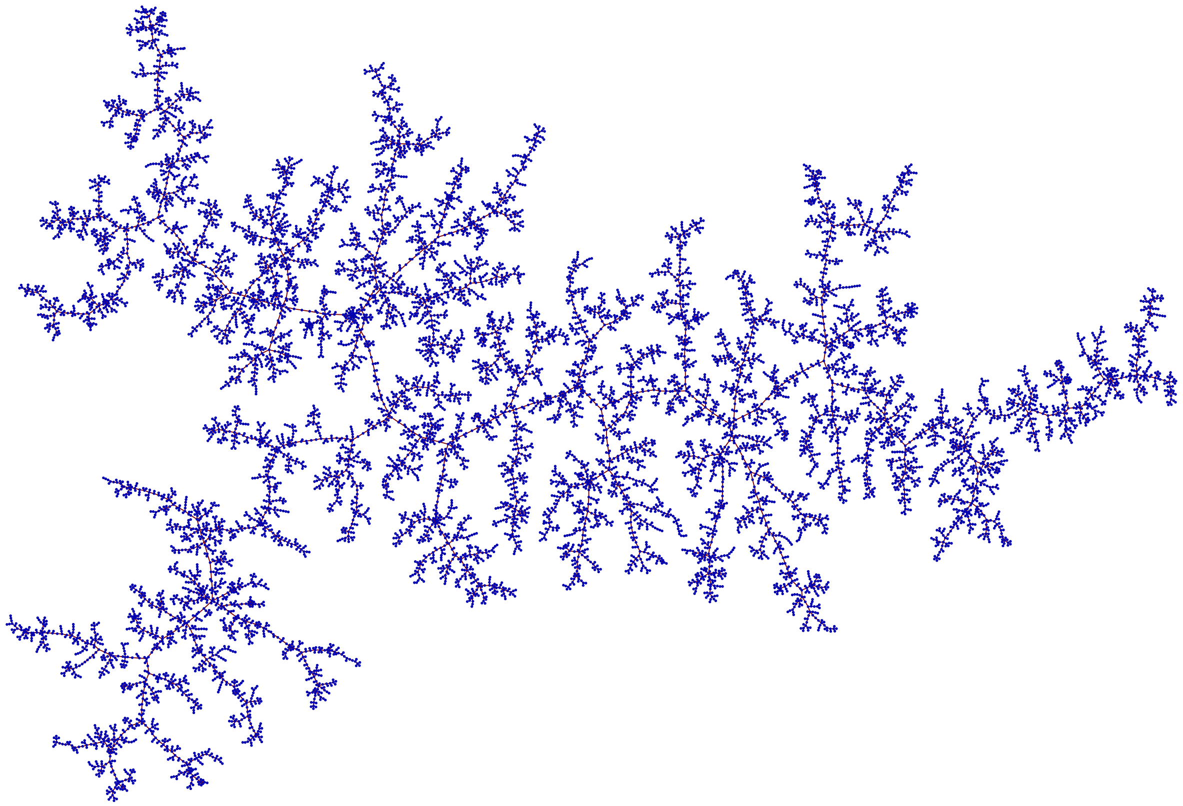
Limits of large random discrete structures





Igor Kortchemski

CNRS & École polytechnique

GRAAL summer school – CIRM – July 2019

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$\leadsto$ Find the cardinal of $\mathcal{X}_n$. (bijective methods, generating functions)

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⇝ Understand the typical properties of $\mathcal{X}_n$.

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- Understand the typical properties of $\mathcal{X}_n$. Let X_n be an element of $\mathcal{X}_n$ chosen *uniformly at random.* What can be said of X_n?
- A possibility to study X_n is to find a limiting object X such that $X_n \to X$ as $n \to \infty$.

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- *From the world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.
- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

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- *In what space do the objects live?* Here, a metric space (Z, d) which will be complete and separable (there exists a dense countable subset).
- *What is the sense of the convergence when the objects are random?* Here, convergence in distribution:

$$\mathbb{E}\left[F(X_n)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[F(X)\right]$$

for every continous bounded function $F : Z \to \mathbb{R}$.

Outline

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Stack triangulations (Albenque, Marckert)

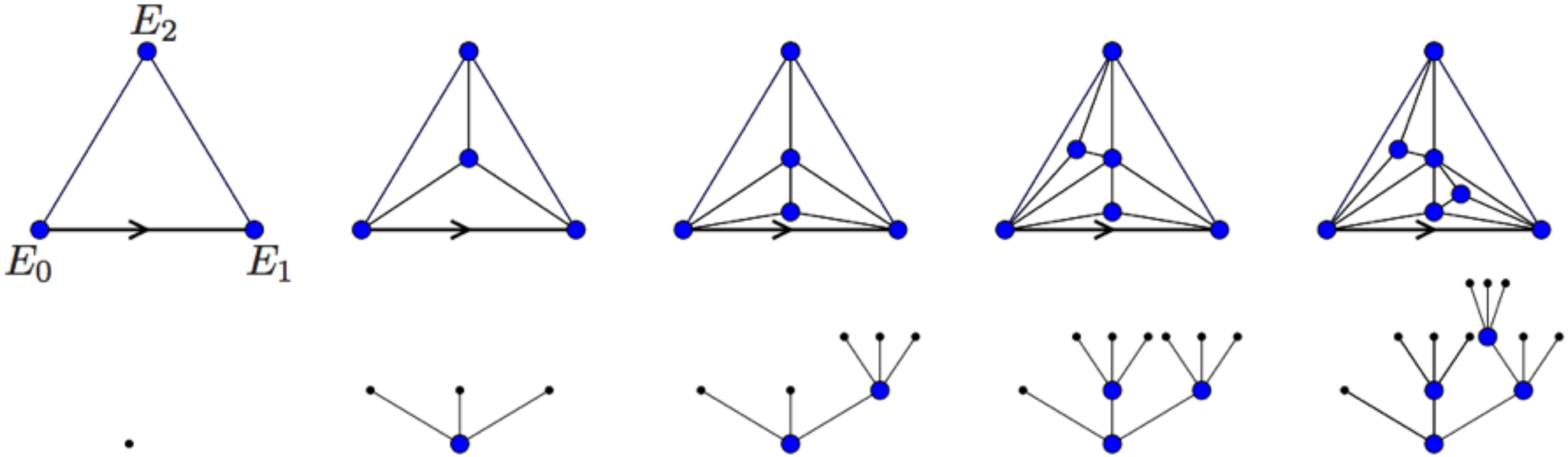


Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

Dissections (Curien, K.)

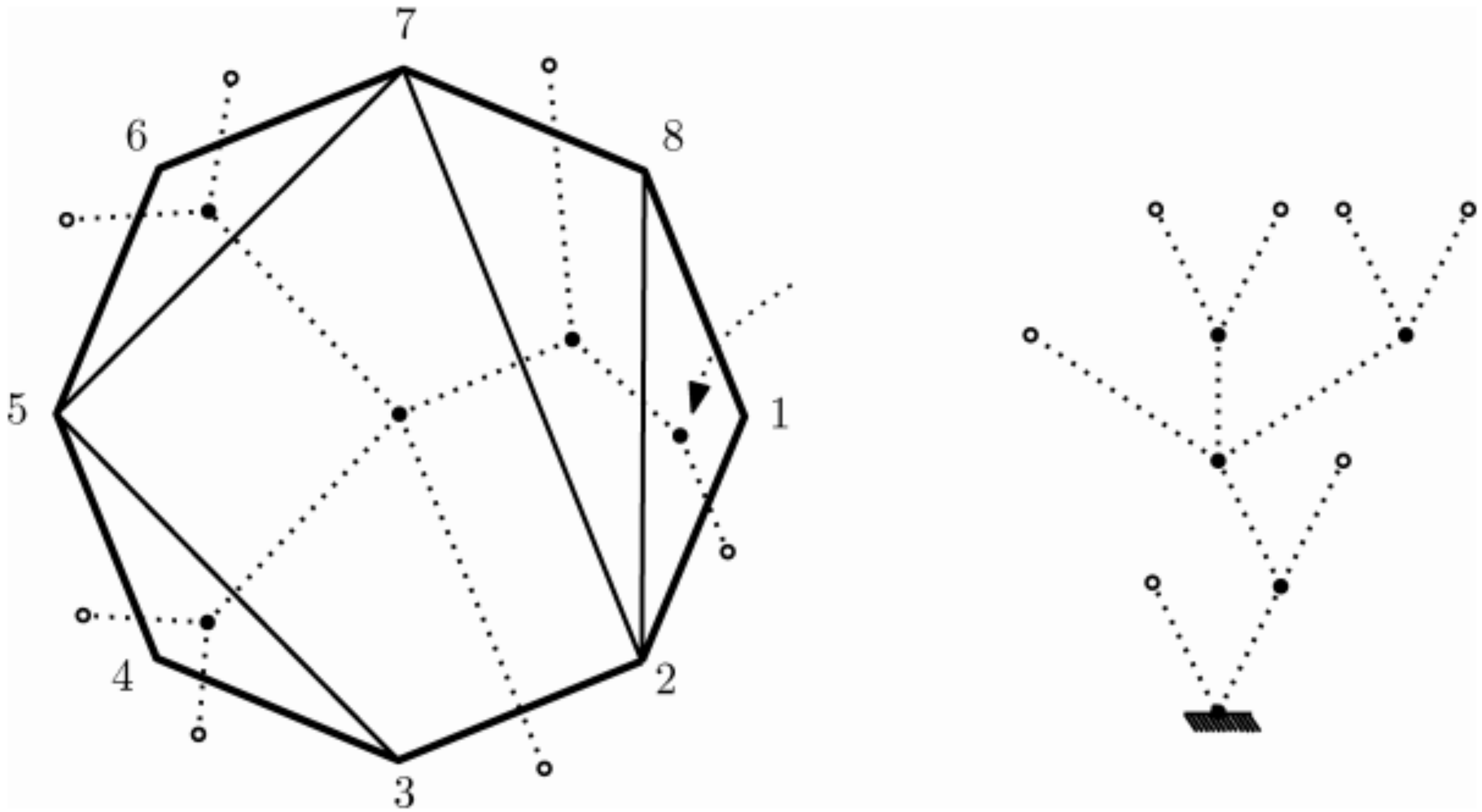


Fig. 4. The dual tree of a dissection of P_8, note that the tree has 7 leaves.

Maps (Schaeffer)

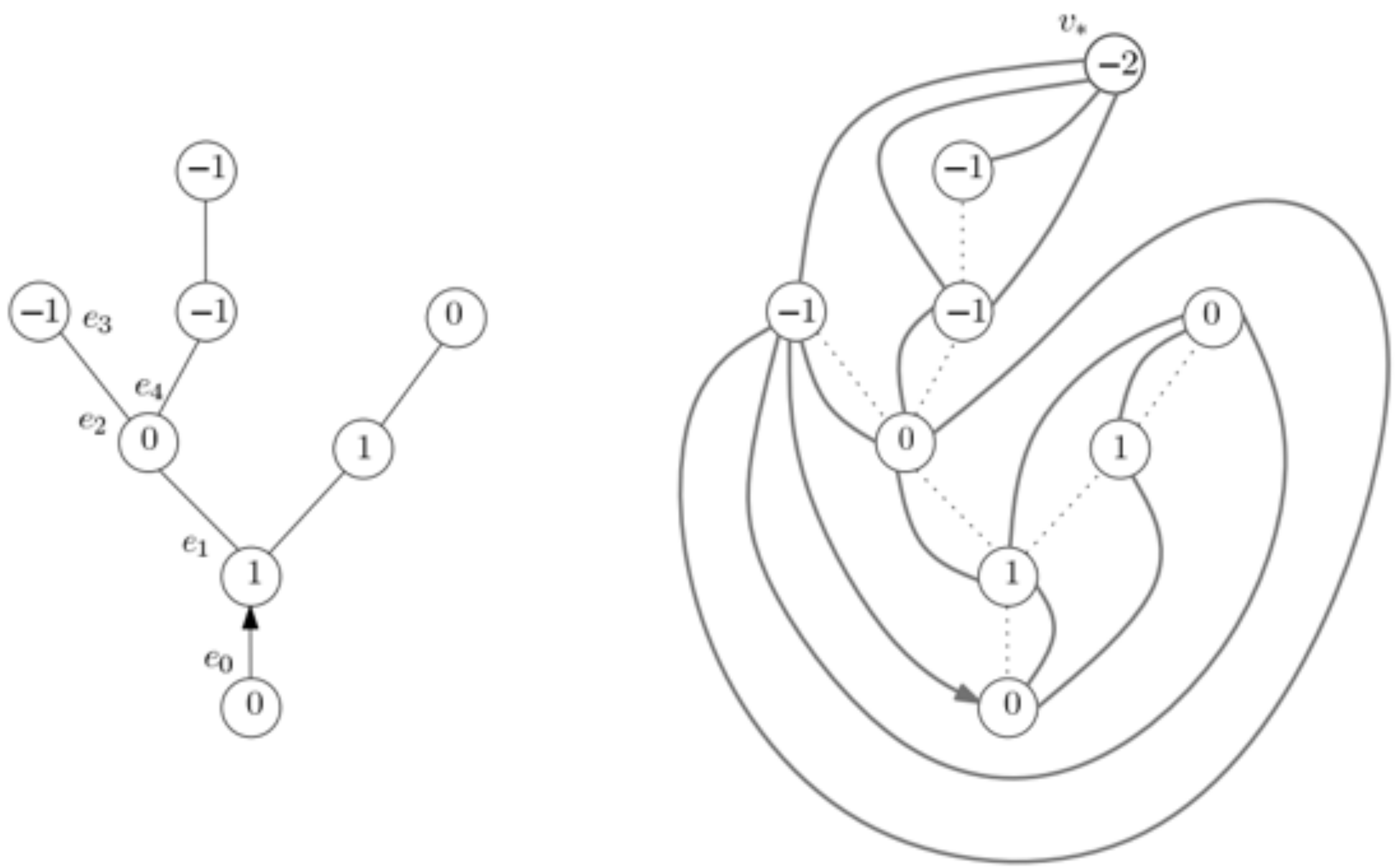
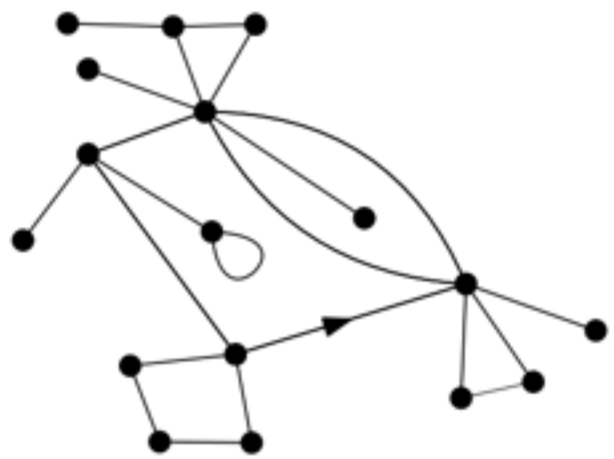
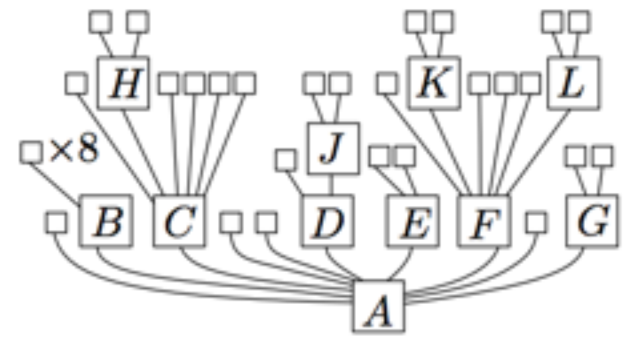


FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

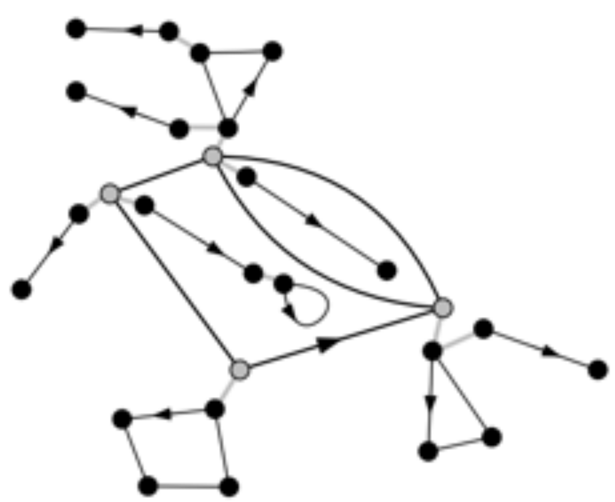
Maps (Addario-Berry)



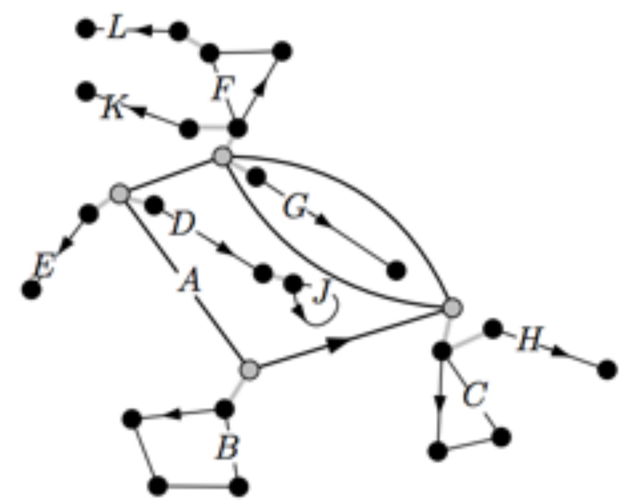
(A) A map M.



(B) The tree T_M. Tiny squares represent trivial blocks.

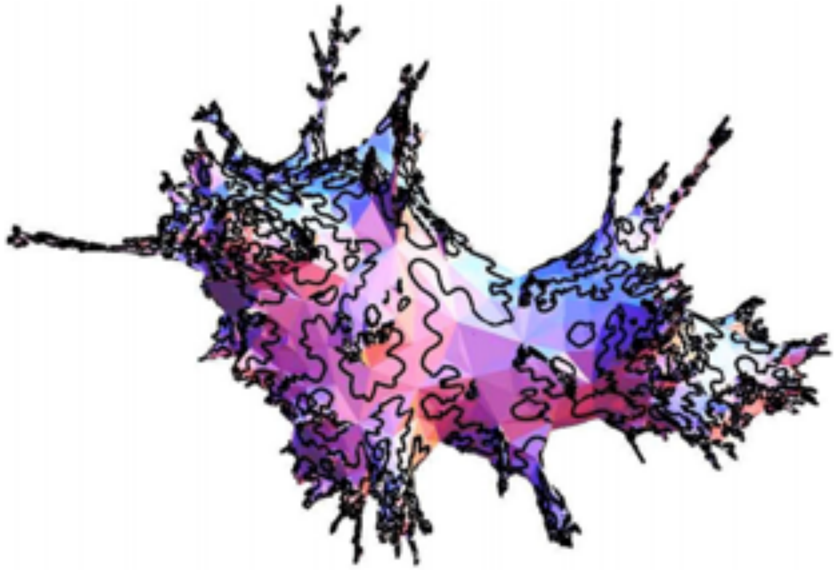


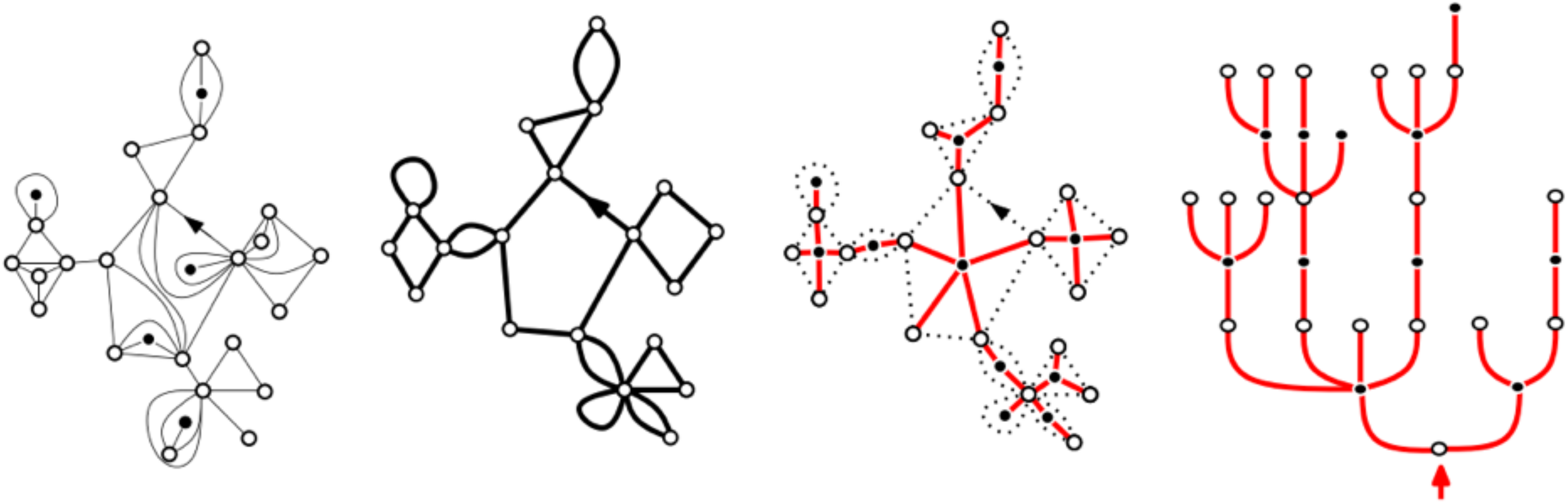
(C) The decomposition of M into blocks. Blocks are joined by grey lines according to the tree structure. Root edges of blocks are shown with arrows.



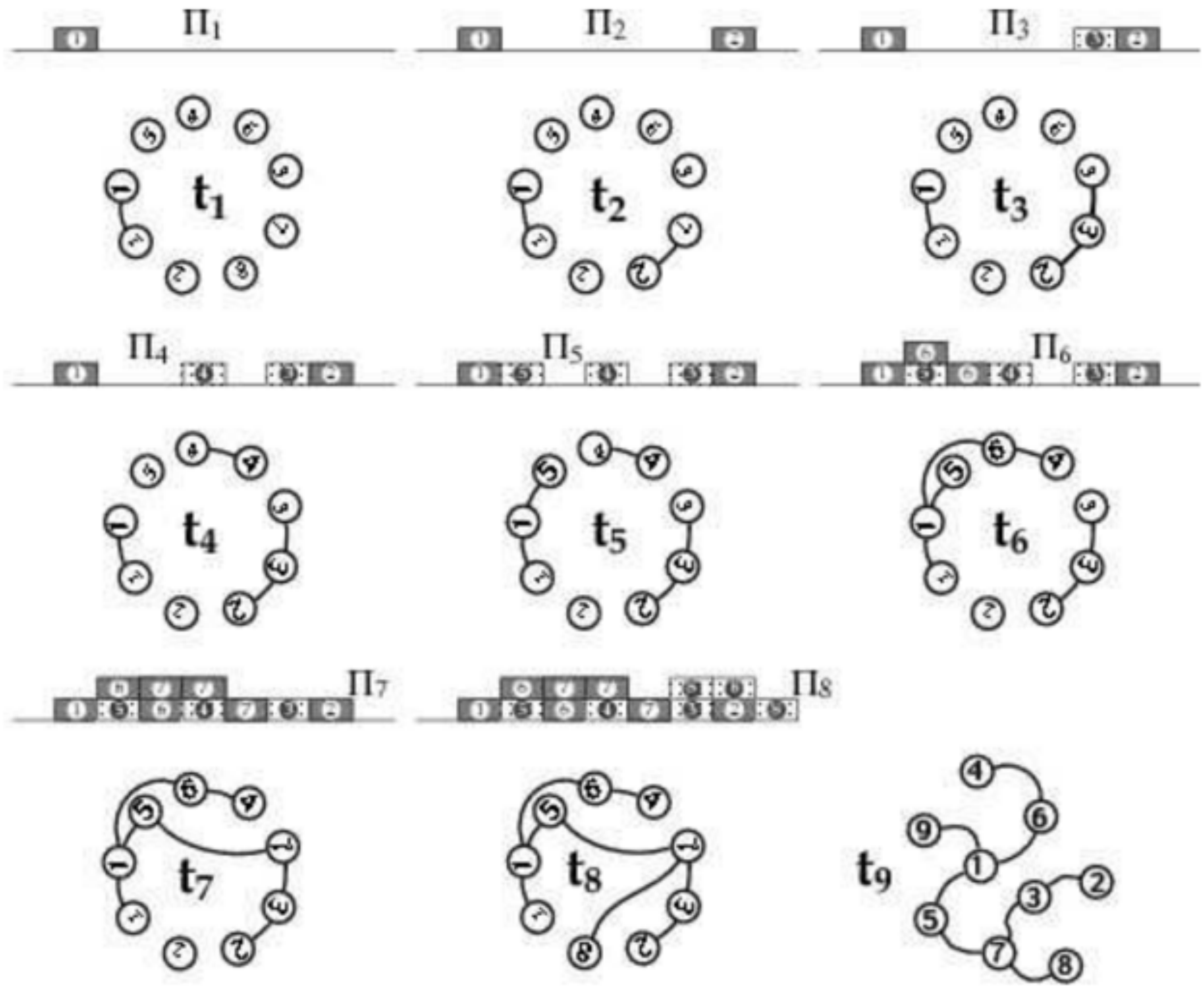
(D) The correspondence between blocks and nodes of T_M. Non-trivial blocks receive the alphabetical label (from A through L) of the corresponding node.

Maps with percolation (Curien, K.)





Parking functions (Chassaing, Louchard))



I. Models coded by trees

II. Local limits of BGW trees

III. Scaling limits of BGW trees

Recall that in a BGW tree, every individual has a random number of children (independently of each other) distributed according to μ (offspring distribution).

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What does a large BGW tree look like, near the root?

Local limits: critical case

Let μ be a **critical** offspring distribution. Let $\mathcal{T}_n$ be a BGW tree conditioned on having n vertices.

Theorem (Kesten '87, Janson '12, Abraham & Delmas '14)

The convergence

$$\mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_\infty$$

holds in distribution for the local topology, where $\mathcal{T}_\infty$ *is the infinite BGW tree conditioned to survive.*

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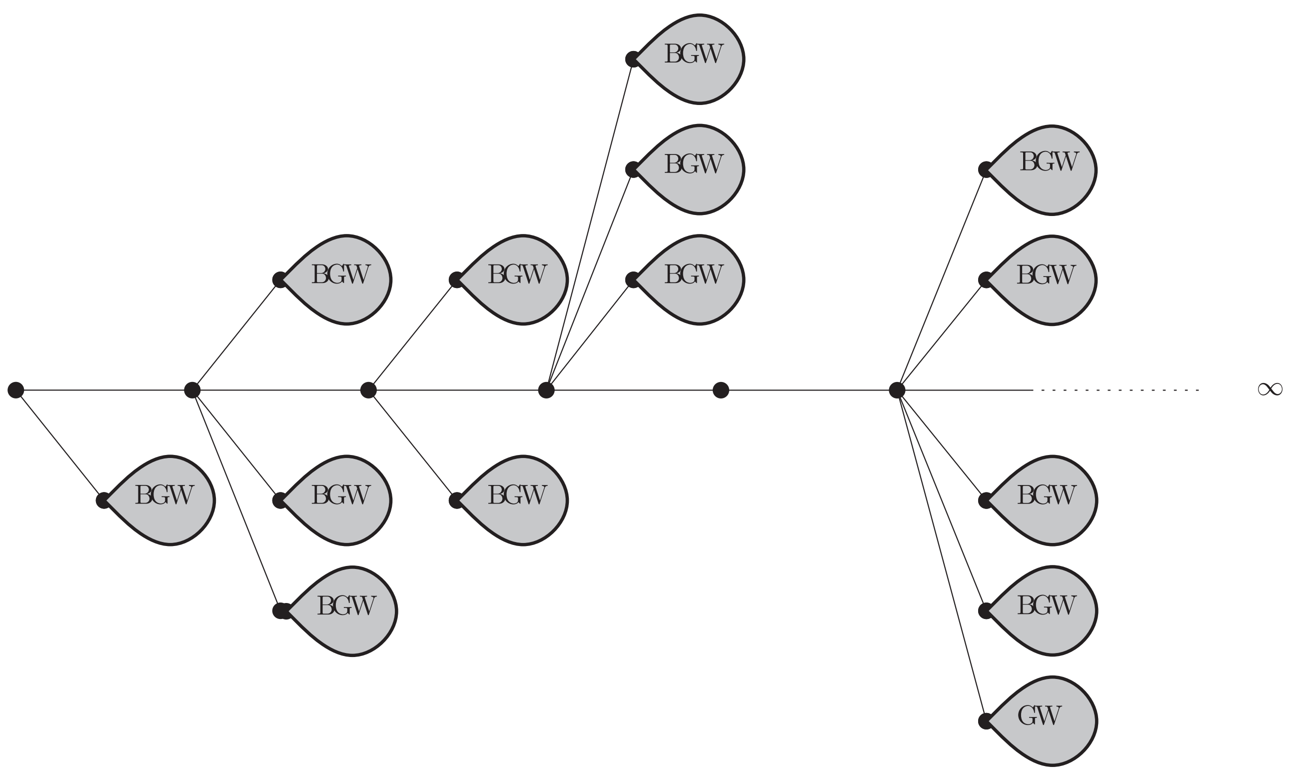

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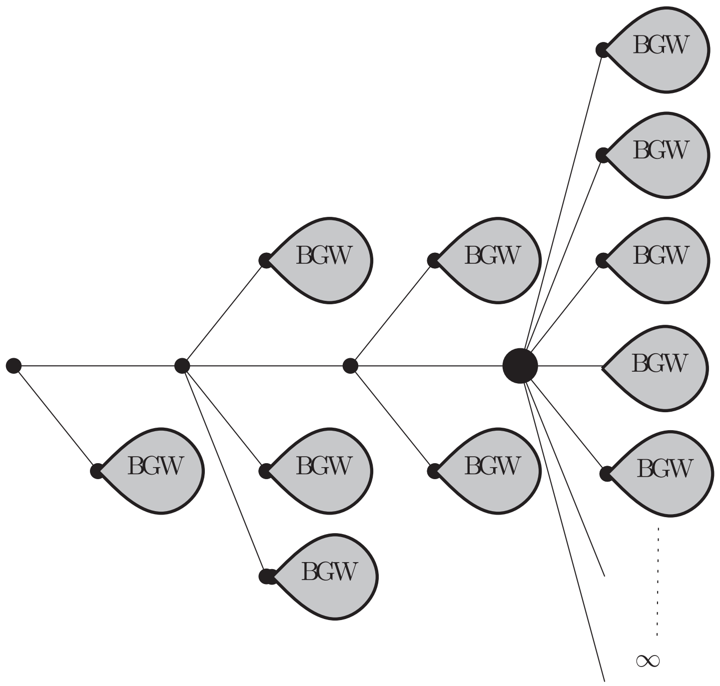

I. MODELS CODED BY TREES

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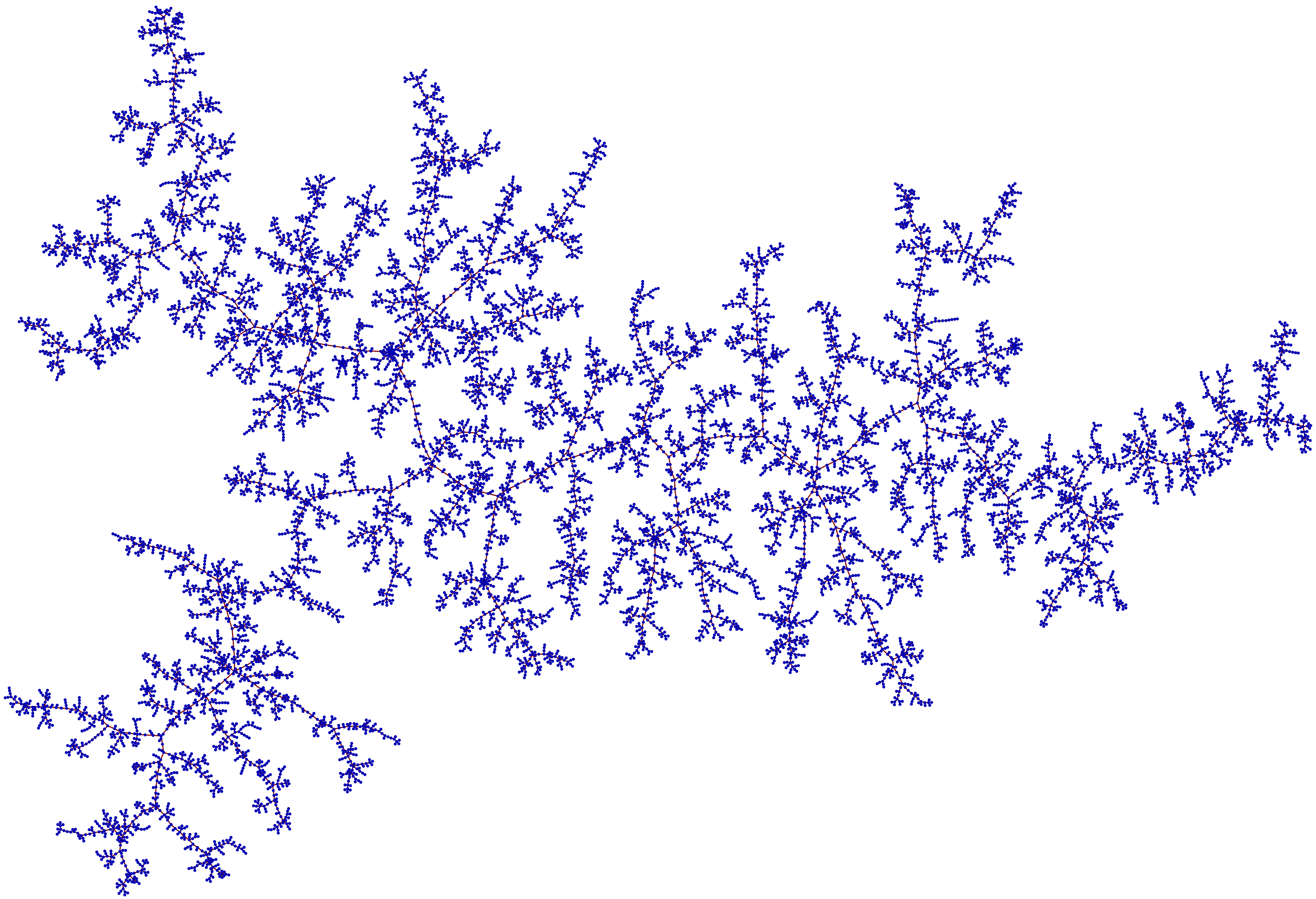
What does a large BGW tree look like, globally?

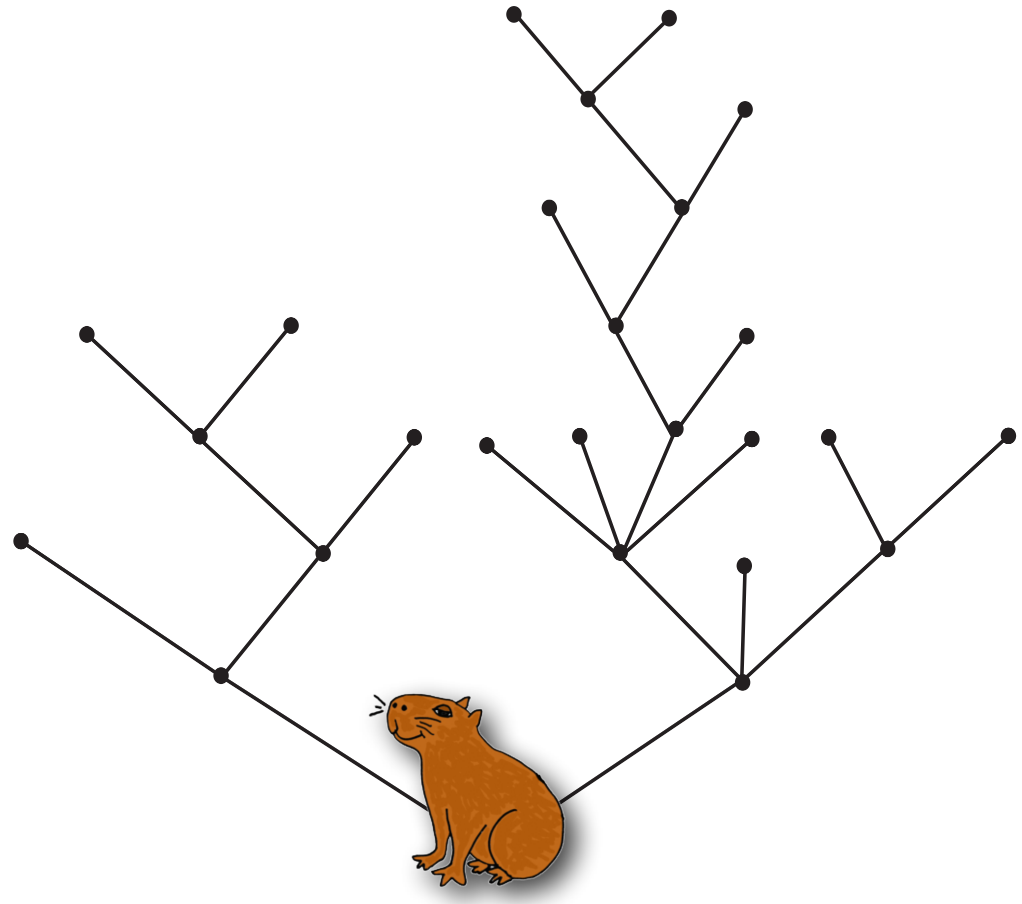
A simulation of a large random critical GW tree

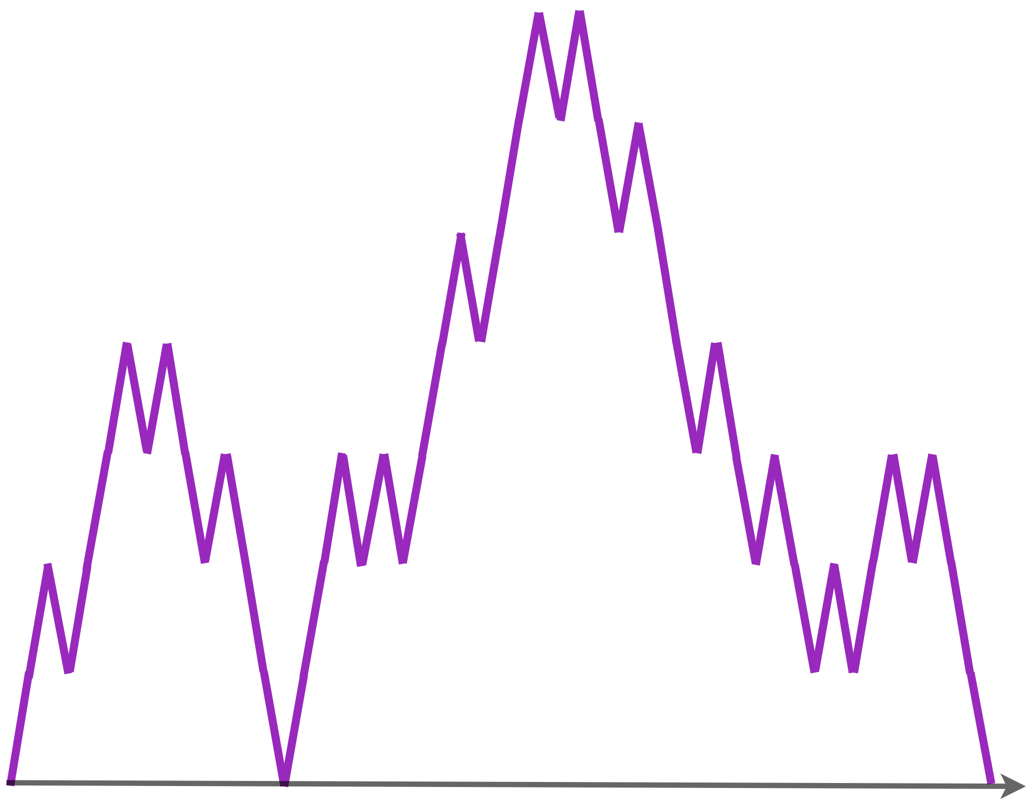


Coding trees by functions

Contour function of a tree

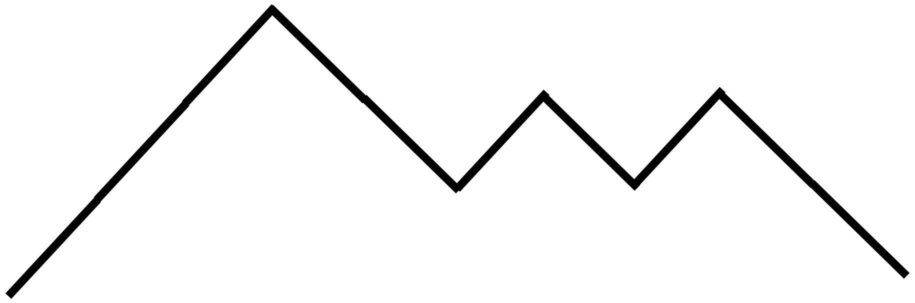
Define the contour function of a tree:





Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



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Let σ^2 be the variance of μ. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n)\right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n \to \infty]{(d)}$$

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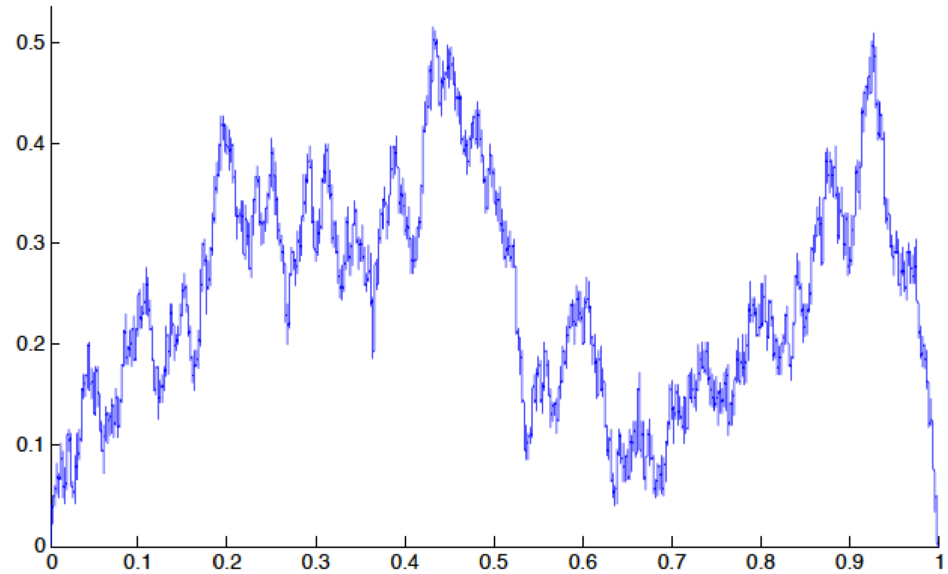

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- Go from the Lukasieiwicz path of $\mathcal{T}_n$ to its contour function.

Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

The Hausdorff distance

Let X, Y be two subsets of the **same** metric space Z.

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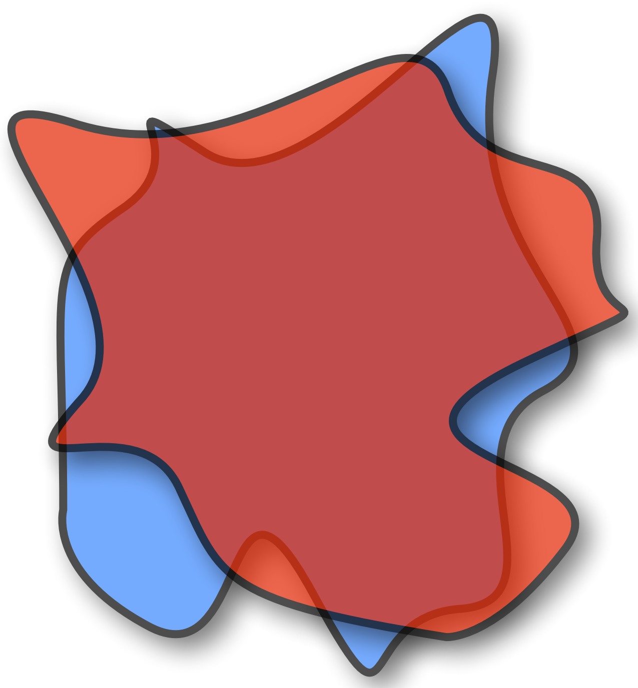

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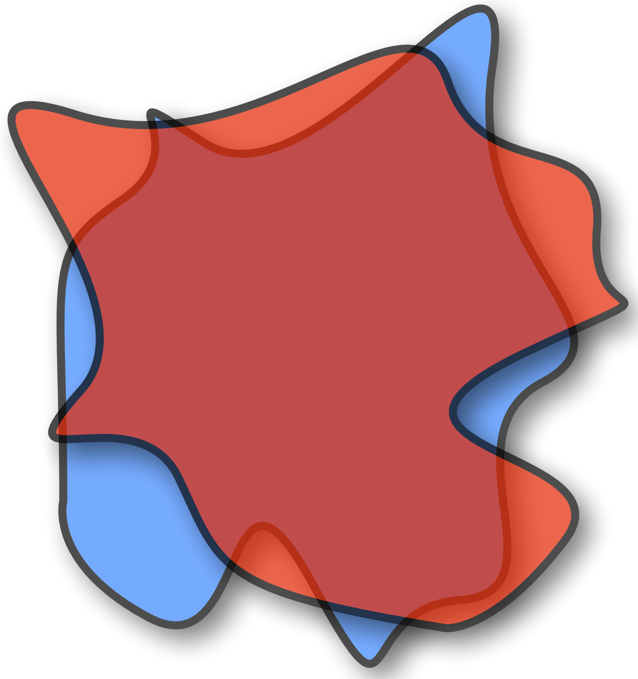

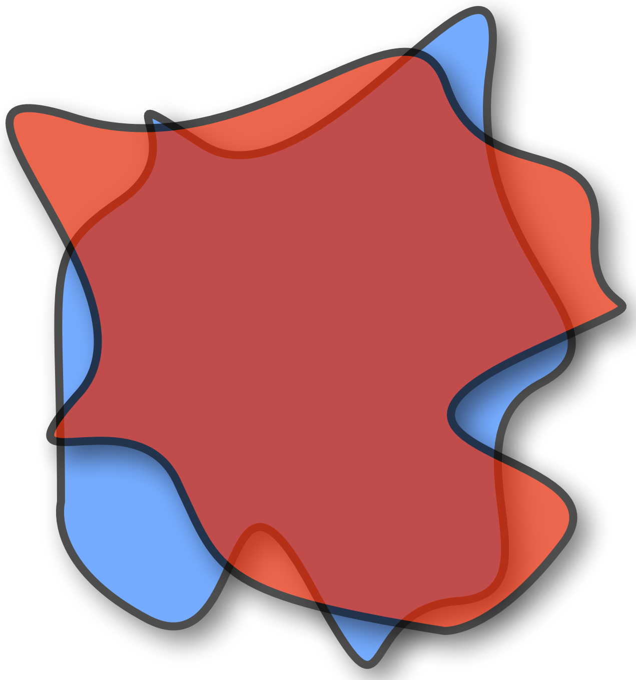

The Gromov–Hausdorff distance

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The Gromov–Hausdorff distance

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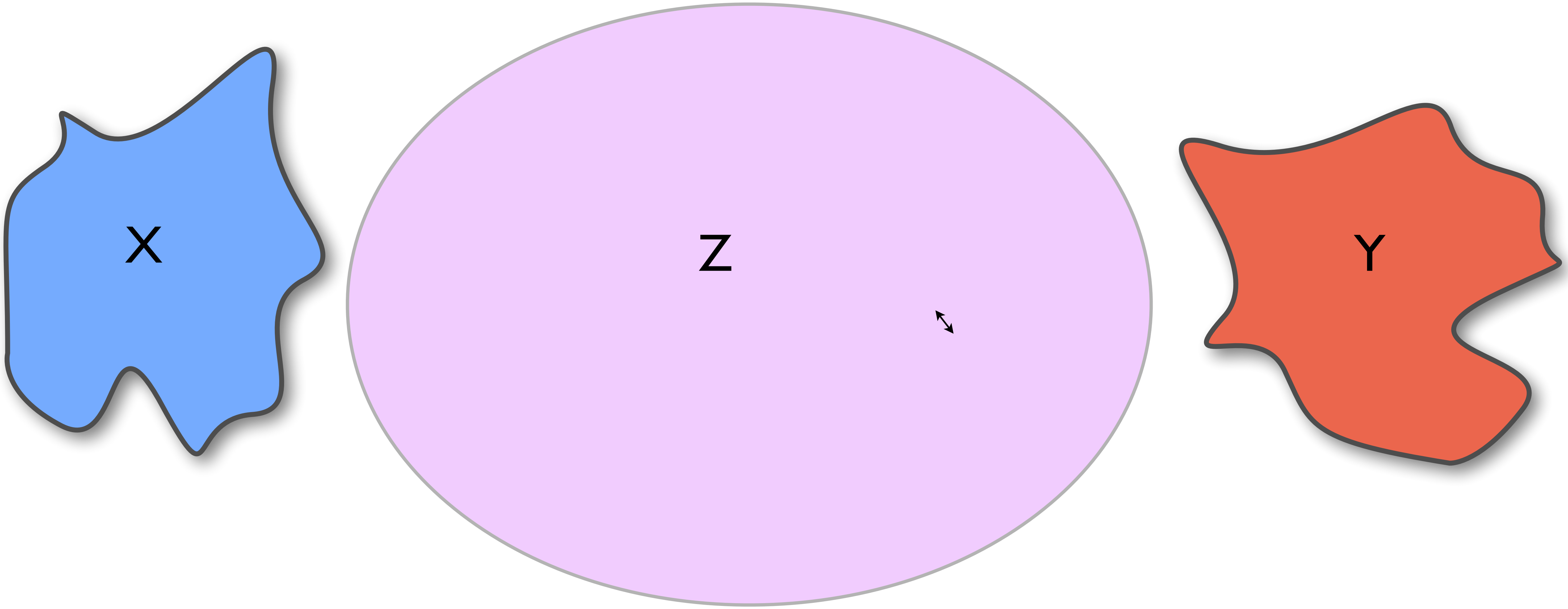

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

$\rightsquigarrow$ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

$$\frac{\sigma}{2\sqrt{n}} \cdot \mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_{\mathbb{e}},$$

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The metric space $\mathcal{T}_{\mathbb{e}}$ is called the *Brownian continuum random tree (CRT)*, and is coded by a Brownian excursion.

Scaling limits: infinite variance case

Scaling limits: domain of attraction of a stable law

Fix $\alpha \in (1,2)$. Let μ be an offspring distribution such that

$$\sum_{i \geqslant 0} i\mu_i = 1 \qquad (\mu \text{ is critical})$$

$$\mu_i \underset{i \to \infty}{\sim} \frac{c}{i^{1+\alpha}} \qquad (\mu \text{ has a heavy tail})$$

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What does $\mathcal{T}_n$ look like for large n?

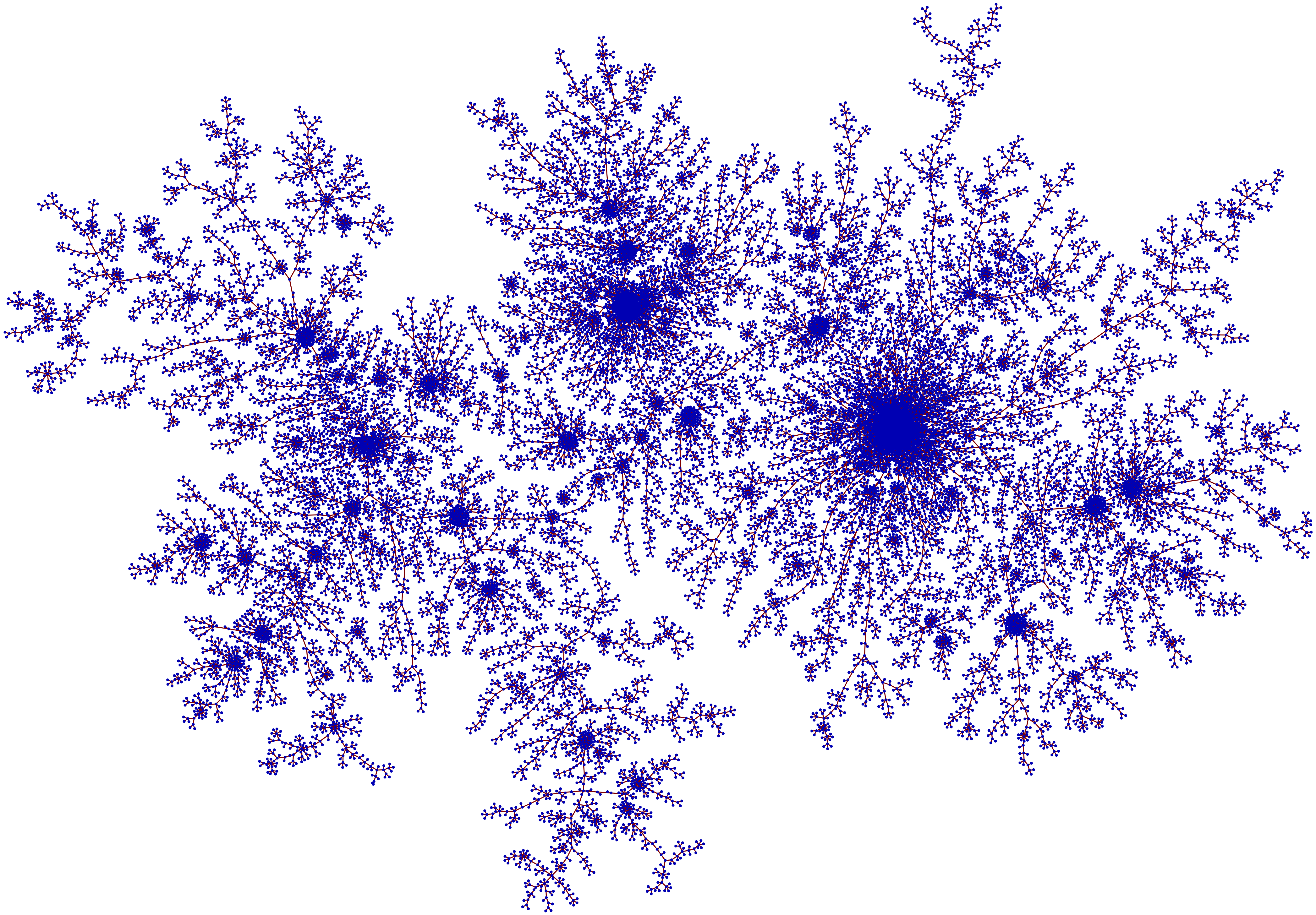

Figure: A large $\alpha = 1.1$ – stable tree

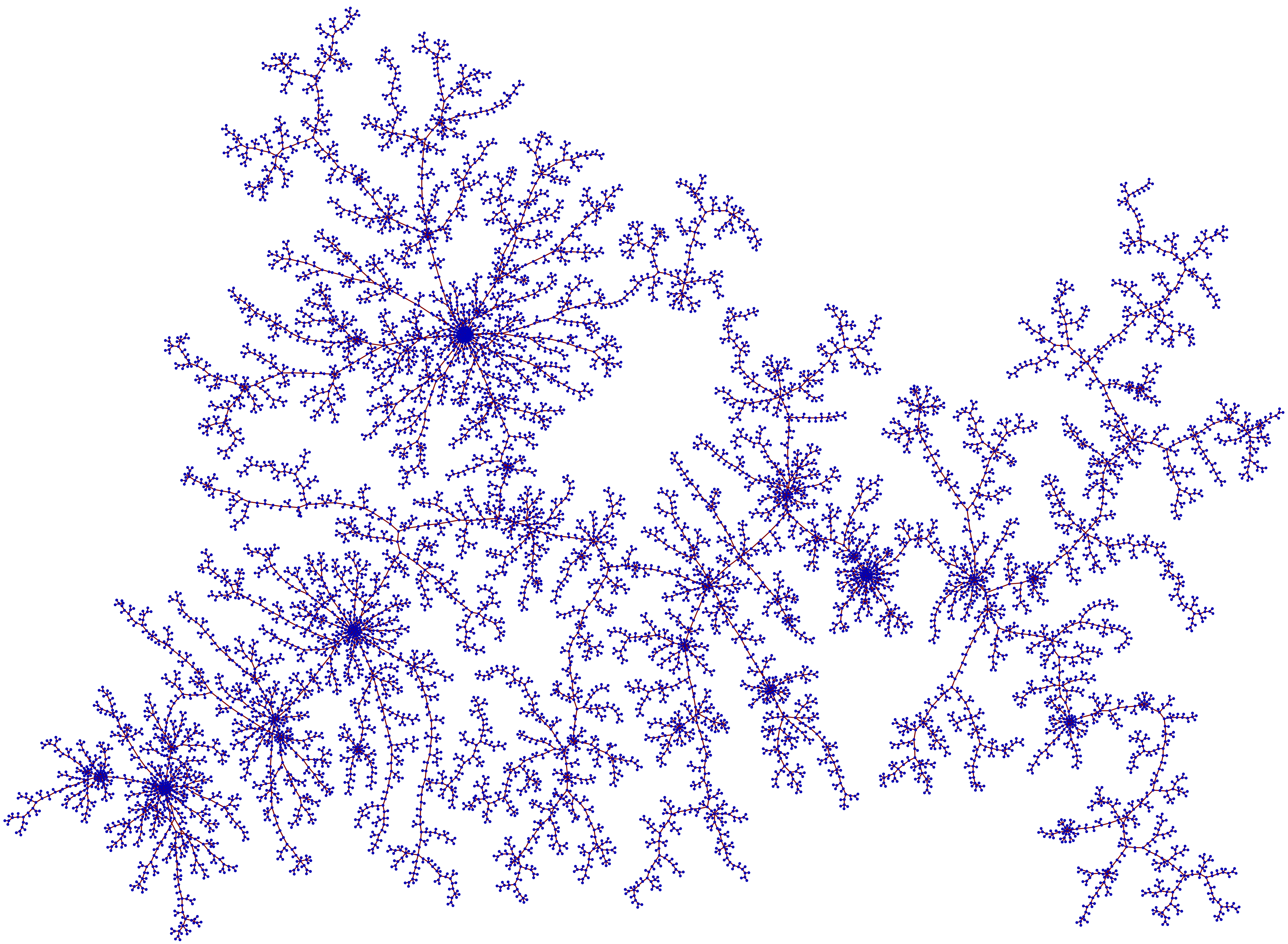

Figure: A large $\alpha = 1.5$ – stable tree

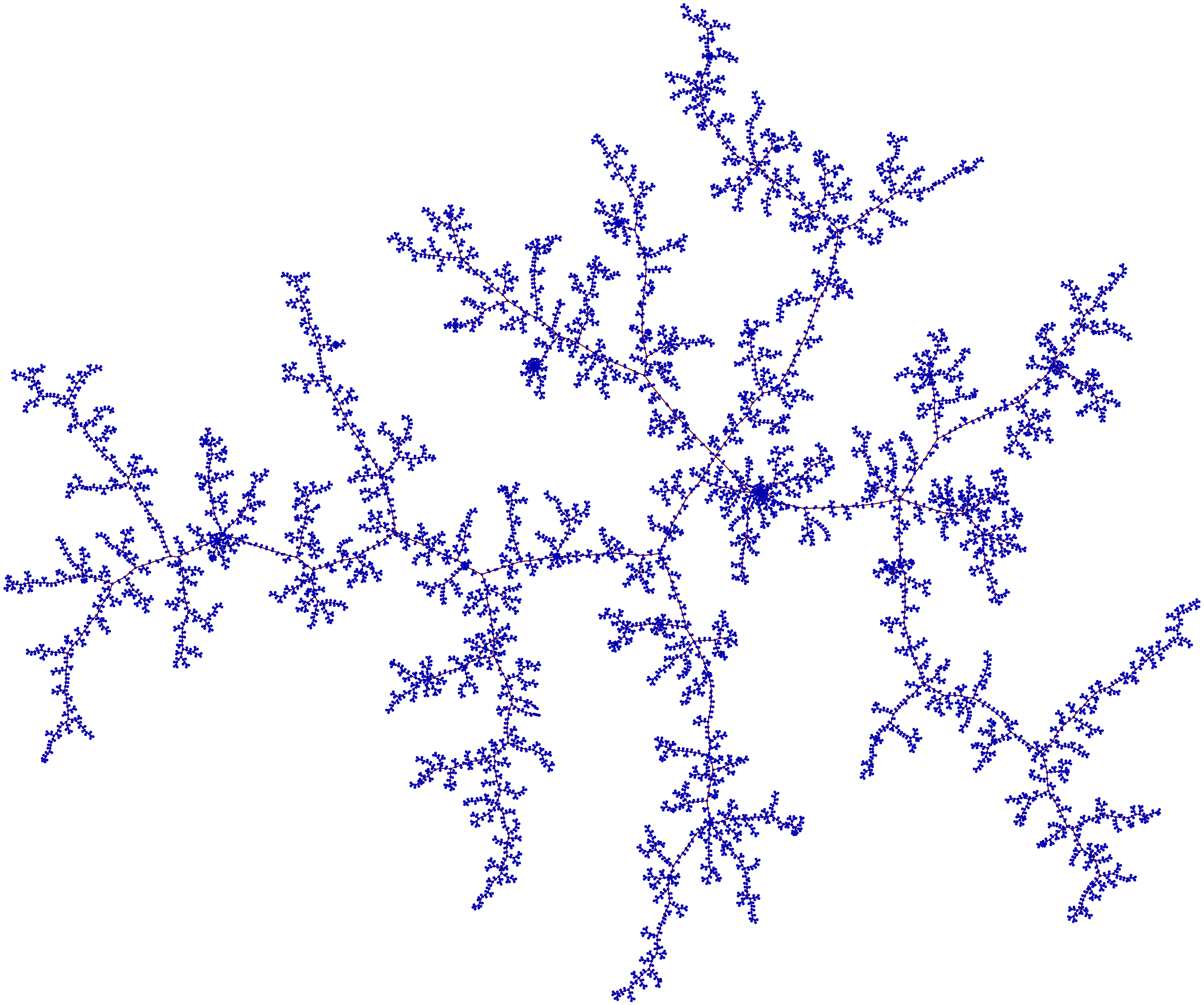

Figure: A large $\alpha = 1.9$ – stable tree

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There exists a random compact metric space $\mathcal{T}_\alpha$ such that:

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⇝ The maximal degree of $\mathcal{T}_n$ is of order $n^{1/\alpha}$.

Condensation : subcritical case

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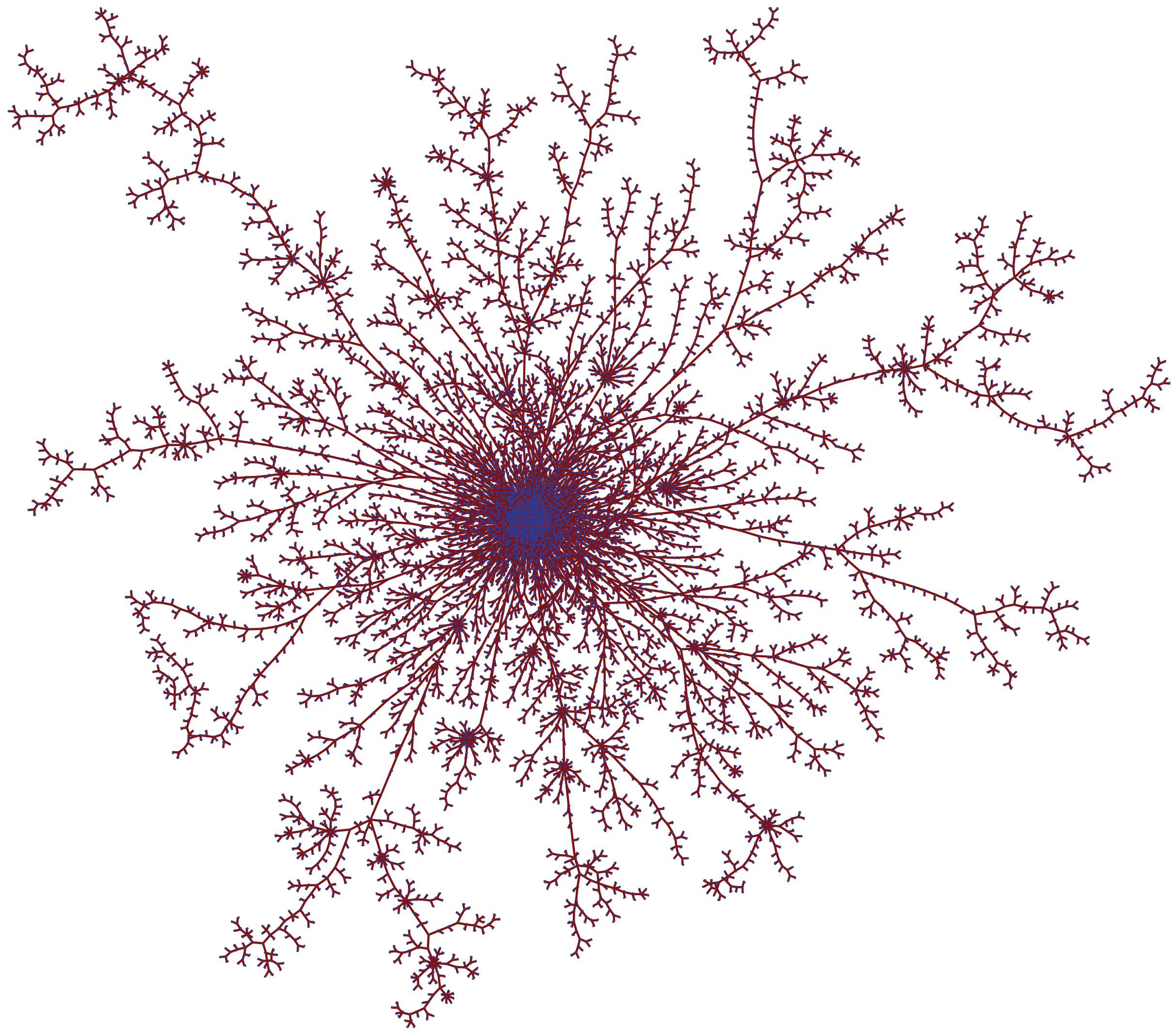

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- there are no nontrivial scaling limits.

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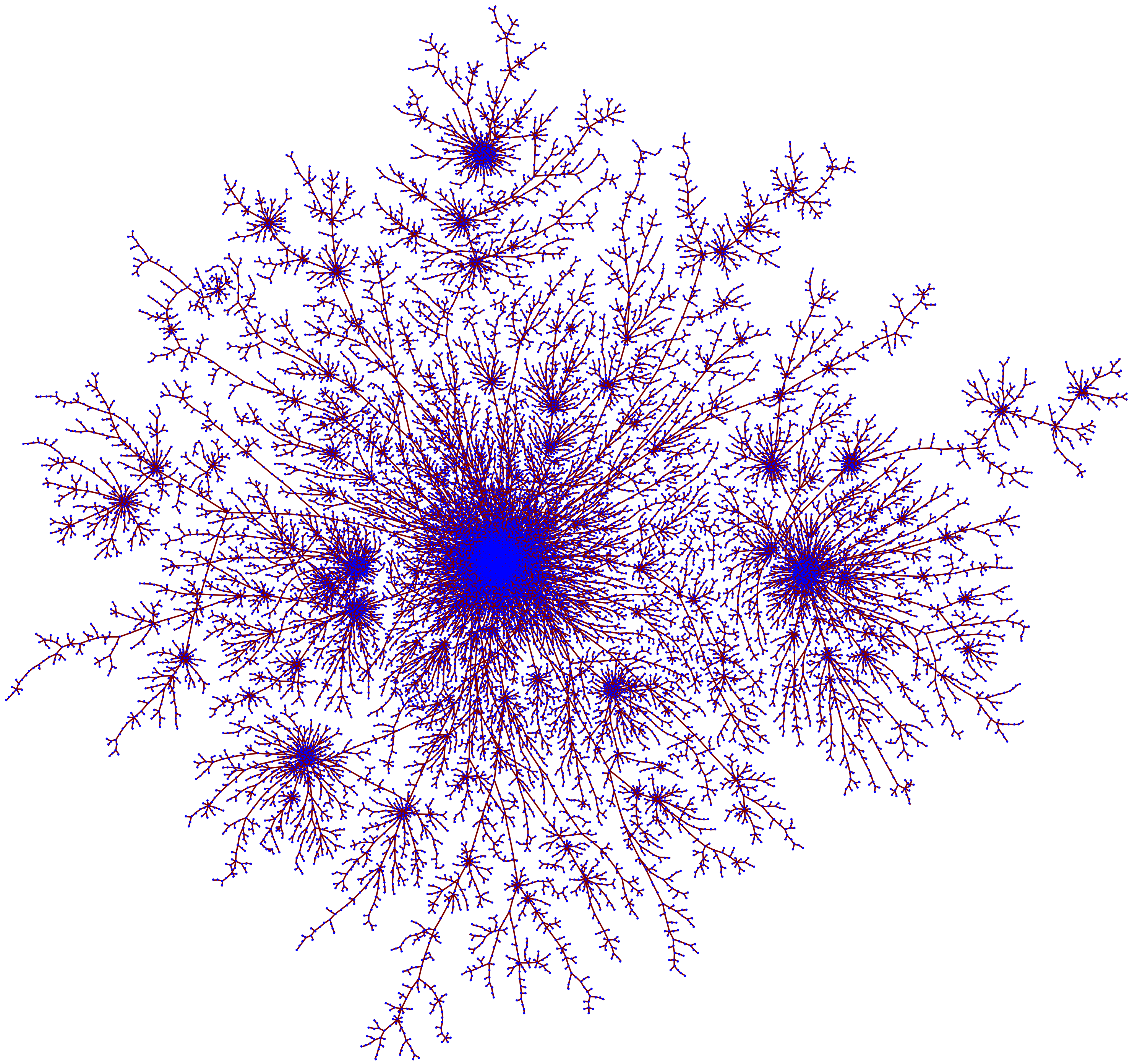

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For example, if

$$\mu_i \sim \frac{1}{\ln(i)^2 i^2},$$

the maximal degree is of order $n/\ln(n)$, the maximum of the other degrees is of order $n/\ln(n)^2$, and the height of the vertex with maximal degree is of order $\ln(n)$.

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- μ *is subcritical and* $\mu(n) = L(n)/n^{1+\beta}$ *with* $\beta > 1$ *and* L *slowly varying.*
  Then condensation occurs: there is a unique vertex of degree of order n (up to a constant), the other degrees are of order $n^{1/\min(2,\beta)}$ (up to a slowly varying constant), the height of the vertex with maximal degree converges in distribution, the height of the tree is of order $\ln(n)$ and there are no nontrivial scaling limits.

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- μ *is critical and* $\mu(n) = L(n)/n^2$ *with* L *slowly varying.*
  Condensation occurs, but at a smaller scale, that is $n/L_1(n)$ (where L_1 is slowly varying), the other degrees are of order $n/L_2(n)$ (where L_2 is slowly varying, with $L_2 = o(L_1)$), and the height of the vertex with maximal degree converges in probability to ∞.