Limit theorems for conditioned non-generic Galton-Watson trees

Igor Kortchemski (Université Paris-Sud, Orsay)
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**Typical framework:**
- the offspring distribution $\mu$ is critical ($\sum_{i \geq 0} i \mu(i) = 1$).
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What happens when $\mu$ is not critical?
Outline

I. State of the art (critical case)

II. Non-generic trees

III. Limit theorems for non-generic trees

IV. One conjecture and one problem
I. State of the art
Recap on Galton-Watson trees

Trees will be planar and rooted.
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Let $\rho$ be a probability measure on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with $\sum_i i \rho(i) \leq 1$ and $\rho(1) < 1$. 
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Let \( \rho \) be a probability measure on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) with \( \sum_i i \rho(i) \leq 1 \) and \( \rho(1) < 1 \). A Galton-Watson tree with offspring distribution \( \rho \) is a random tree such that:

1. \( k^\emptyset \) has distribution \( \rho \), where \( k^\emptyset \) is the number of children of the root.
2. for every \( j \geq 1 \) with \( \rho(j) > 0 \), under \( P^\rho(\cdot | k^\emptyset = j) \), the number of children of the \( j \) children of the root are independent with distribution \( \rho \).
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Condensation in Galton-Watson trees$
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Here, $k_\emptyset = 2$. 
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Let $\zeta(\tau)$ denote the total number of vertices of $\tau$. 
Scaling limits
Coding trees
Order the vertices in the *lexicographical order*:

\[ k_{\emptyset} = u(0) < u(1) < \cdots < u(\zeta(\tau) - 1). \]

Let \( k_u \) be the number of children of the vertex \( u \).
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Definition
The Lukasiewicz path \( W(\tau) = (W_n(\tau), 0 \leq n \leq \zeta(\tau)) \) of a tree \( \tau \) is defined by:

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Proposition

The Lukasiewicz path of a $GW_\mu$ tree has the same distribution as a random walk with jump distribution $\nu(k) = \mu(k + 1)$, $k \geq -1$, started from 0, stopped when it hits $-1$. 
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**Definition (of the contour function)**

A capybara explores the tree at unit speed. For \( 0 \leq t \leq 2(\zeta(\tau) - 1) \), \( C_t(\tau) \) is the distance between the beast at time \( t \) and the root.
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Figure: The Lukasiewicz path and the contour function.

- The Lukasiewicz path behaves like a random walk.
Scaling limits

Let $\mu$ be a critical offspring distribution with finite variance. Let $t_n$ be a $P_{\mu} [\cdot | \zeta(\tau) = n]$ tree. What does $t_n$ look like for $n$ large?
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**Theorem (Aldous ’93, Duquesne ’04)**

Let $\sigma^2$ be the variance of $\mu$. Then:

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\left( \frac{1}{\sqrt{n}} W_{[nt]}(t_n), \frac{1}{2\sqrt{n}} C_{2nt}(t_n) \right)_{0 \leq t \leq 1} \overset{(d)}{\underset{n \to \infty}{\longrightarrow}} \left( \sigma \cdot e(t), \frac{1}{\sigma} e(t) \right)_{0 \leq t \leq 1},
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where $e$ is the normalized Brownian excursion.
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**Remark:**

- Duquesne '04: extension to the case where $\mu$ is in the domain of attraction of a stable law.
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**Consequences:**

- limit theorem for the height of $t_n$,
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**Consequences:**
- limit theorem for the height of $t_n$,
- convergence in the Gromov-Hausdorff sense of $t_n$, suitably rescaled, towards the Brownian CRT.
II. Non-generic trees
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Exponential families

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**Lemma (Kennedy ’75)**

Let $\lambda > 0$ be such that

$$Z_\lambda = \sum_{i \geq 0} \mu(i)\lambda^i < \infty.$$

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Then a $GW_\mu$ tree **conditioned** on having $n$ vertices has the same distribution as a $GW_{\mu^{(\lambda)}}$ tree **conditioned** on having $n$ vertices.
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Then a $GW_\mu$ tree **conditioned** on having $n$ vertices has the same distribution as a $GW_{\mu^{(\lambda)}}$ tree **conditioned** on having $n$ vertices.

**Consequence:**

- if there exists $\lambda > 0$ such that $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical, then we are back to the critical case.
Definition

We say that $\mu$ is **non-generic** if there exist no $\lambda > 0$ such that $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical.
Exponential families

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Example:
- $\mu$ is subcritical ($\sum_i i \mu(i) < 1$)
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- $\mu$ is subcritical ($\sum_i i \mu(i) < 1$)
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Example: $\mu(i) \sim c/i^\beta$ with $c > 0$ and $\beta > 2$. 
II. 2) **Large non-generic trees**
Large non-generic trees

Fix $\mu$ non-generic. What does a $\mathbb{P}_\mu [\cdot | \zeta(\tau) = n]$ tree look like for $n$ large (Jonsson & Stefánsson 11’)?
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Condensation phenomenon
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Condensation phenomenon (which also appears in the zero-range process !).
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1) Then:

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1) Then:

$$t_n \xrightarrow{(d)} \hat{T}$$

where the convergence holds for the local convergence and the tree $\hat{T}$ has the following form:

![Diagram of a tree structure](image-url)

$S = 4$
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The spine has a finite random length $S$, where:

$$\mathbb{P}[S = i] = (1 - m)m^i \quad \text{for } i \geq 0$$

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1) We have $t_n \xrightarrow{(d)\quad n \to \infty} \hat{T}$ where $\hat{T}$ is:

![Diagram of tree with GW nodes and S = 4]

$\mathbb{P}(S = i) = (1 - m)m^i$
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1) We have $t_n \xrightarrow{d} n \to \infty \hat{T}$ where $\hat{T}$ is:

2) The maximal degree of $t_n$, divided by $n$, converges in probability towards $1 - m$. 

Questions:
- Do the Lukasiewicz path and contour function of $t_n$, properly rescaled, converge?
- What are the fluctuations of the maximal degree?
- Where is located the vertex of maximal degree?
- What is the height of $t_n$?
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Remarks:
- In the critical case the spine is infinite (Kesten ’86).
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- Janson ’12: Assertion 1) holds for every non-generic $\mu$. 
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- In the critical case the spine is infinite (Kesten ’86).
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- A \( GW_\mu \) tree has in expectation \( 1 + m + m^2 + \cdots = 1/(1 - m) \) vertices.
**Theorem (Jonsson & Stefánsson '11)**

Let $\mu$ be a subcritical offspring distribution such that $\mu(i) \sim c/i^\beta$ with $c > 0$, $\beta > 2$. Let $t_n$ be a $P_\mu[\cdot|\zeta(\tau) = n]$ tree and $m$ be the mean of $\mu$.

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**Remarks:**

- In the critical case the spine is infinite (Kesten '86).
- Janson '12: Assertion 1) holds for every non-generic $\mu$.
- A $GW_\mu$ tree has in expectation $1 + m + m^2 + \cdots = 1/(1 - m)$ vertices. Hence a forest of $cn$ trees $GW_\mu$ has in expectation $cn/(1 - m)$ vertices.
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1) We have $t_n \xrightarrow{(d)_{n \to \infty}} \hat{T}$ where $\hat{T}$ is:

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- What are the fluctuations of the maximal degree?
Theorem (Jonsson & Stefánsson ’11)

Let $\mu$ be a subcritical offspring distribution such that $\mu(i) \sim c/i^{\beta}$ with $c > 0$, $\beta > 2$. Let $t_n$ be a $\mathbb{P}_\mu \left[ \cdot \mid \zeta(\tau) = n \right]$ tree and $m$ be the mean of $\mu$.

1) We have $t_n \xrightarrow{d} \hat{T}$ where $\hat{T}$ is:

2) The maximal degree of $t_n$, divided by $n$, converges in probability towards $1 - m$.

Questions:
- Do the Lukasiewicz path and contour function of $t_n$, properly rescaled, converge?
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- What are the fluctuations of the maximal degree?
- Where is located the vertex of maximal degree?
- What is the height of $t_n$?
III. Limit theorems for non-generic trees
Assumptions

We consider an offspring distribution $\mu$ such that:
- $\mu$ is subcritical ($0 < \sum_i i \mu(i) < 1$)
- There exists a slowly varying function $L$ such that $\mu(n) = L(n) n^{-1} + \theta, n \geq 1$ with fixed $\theta > 1$.

$L$ is slowly varying if $L(tx)/L(x) \to 1$ when $x \to \infty$, $\forall t > 0$. 

Let $\tau_n$ be a $P_{\mu}[\cdot | \zeta(\tau) = n]$ tree.
We consider an offspring distribution $\mu$ such that:

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\[
\mu(n) = \frac{L(n)}{n^{1+\theta}}, \quad n \geq 1
\]

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Let $t_n$ be a $\mathbb{P}_\mu [\cdot \mid \zeta(\tau) = n]$ tree.
III. 1) **Convergence of the Lukasiewicz path**
Let \( k_u \) be the number of children of the vertex \( u \).

**Definition**

The Lukasiewicz path \( \mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leq n \leq \zeta(\tau)) \) of a tree \( \tau \) is defined by:

\[
\mathcal{W}_0(\tau) = 0, \quad \mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1.
\]
Convergence of the Lukasiewicz path

Let $U(t_n)$ be the index of the first vertex with maximal degree of $t_n$.

Theorem (K. 12')

(i) $U(t_n)/n$ converges in probability towards 0 as $n \to \infty$.

(ii) $\sup_{0 \leq i \leq U(t_n)} W_i(t_n)n$ $(P)$ $\to_{n \to \infty} 0$.

(iii) $(W_{\lfloor nt\rfloor}(t_n) \vee (U(t_n)+1))(t_n)n, 0 \leq t \leq 1$ $(d)$ $\to_{n \to \infty} ((1-m)(1-t))$ $0 \leq t \leq 1$. 

Igor Kortchemski (Université Paris-Sud, Orsay)
Convergence of the Lukasiewicz path

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**Theorem (K. 12’)**

*We have:*

\[
\frac{W_{t_n}(t_n)}{n} \xrightarrow{n \to \infty} 0.
\]
Let $U(t_n)$ be the index of the first vertex with maximal degree of $t_n$.

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(ii) $\sup_{0 \leq i \leq U(t_n)} \frac{W_i(t_n)}{n} \overset{(P)}{\longrightarrow} 0$.

(iii) $\left( \frac{W_{[nt]} \vee (U(t_n)+1)(t_n)}{n} , 0 \leq t \leq 1 \right) \overset{(d)}{\longrightarrow} ((1-m)(1-t))_{0 \leq t \leq 1}$. 

![Graph](attachment://graph.png)
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**Remarks:**

- The limit is deterministic and depends only on $m$ (the mean of $\mu$).
Convergence of the Lukasiewicz path

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**Remarks:**

- The limit is deterministic and depends only on $m$ (the mean of $\mu$).

- With high probability, there is one vertex with degree roughly $(1 - m)n$ and the others have degree $o(n)$.
We know that $W(t_n)$ has the law of a random walk $(W_n)_{n \geq 0}$ with jump distribution $\nu(k) = \mu(k + 1)$, $k \geq -1$.
We know that $W(t_n)$ has the law of a random walk $(W_n)_{n \geq 0}$ with jump distribution $\nu(k) = \mu(k + 1), k \geq -1$, conditioned on $W_1 \geq 0, W_2 \geq 0, \ldots, W_{n-1} \geq 0$ and $W_n = -1$. But $E[W_1] = m - 1 < 0$. By the "one big jump principle", $W(t_n)$ makes one macroscopic jump, and all the other jumps are asymptotically independent (the distribution of $W_1$ is $(0, 1]$–subexponential).
We know that $\mathcal{W}(t_n)$ has the law of a random walk $(W_n)_{n \geq 0}$ with jump distribution $\nu(k) = \mu(k + 1)$, $k \geq -1$, conditioned on $W_1 \geq 0, W_2 \geq 0, \ldots, W_{n-1} \geq 0$ and $W_n = -1$.

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Idea of the proof

- We know that $\mathcal{W}(t_n)$ has the law of a random walk $(W_n)_{n \geq 0}$ with jump distribution $\nu(k) = \mu(k + 1)$, $k \geq -1$, conditioned on $W_1 \geq 0$, $W_2 \geq 0$, $\ldots$, $W_{n-1} \geq 0$ and $W_n = -1$.

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By the “one big jump principle”, $W(t_n)$ makes one macroscopic jump, and all the other jumps are asymptotically independent (the distribution of $W_1$ is $(0, 1]$–subexponential).
Applications

Let $u_*(t_n)$ be the vertex of maximal degree.
Applications

Let $u_*(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree.
Applications

Let $u_\ast(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u_\ast(t_n)|$ its height.
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Let $u_\star(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u_\star(t_n)|$ its height.

- The fluctuations of $\Delta(t_n)$ around $(1 - m)n$ are of order $n^{2^\theta}$.
Applications

Let $u_\star(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u_\star(t_n)|$ its height.

- The fluctuations of $\Delta(t_n)$ around $(1 - m)n$ are of order $n^{2^\theta}$.
- For $i \geq 0$, $\mathbb{P}[|u_\star(t_n)| = i] \xrightarrow{n \to \infty} (1 - m)m^i$. 

This is not an immediate consequence of the local convergence! 

Recall the local convergence of $t_n$ to Igor Kortchemski (Université Paris-Sud, Orsay) 

Condensation in Galton-Watson trees
Applications

Let $u^*(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u^*(t_n)|$ its height.

- The fluctuations of $\Delta(t_n)$ around $(1 - m)n$ are of order $n^{2^\theta}$.
- For $i \geq 0$, $\mathbb{P}[|u^*(t_n)| = i] \xrightarrow{n \to \infty} (1 - m)m^i$.

Recall the local convergence of $t_n$ to

\[
\mathbb{P}(S = i) = (1 - m)m^i
\]
Applications

Let $u_*(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u_*(t_n)|$ its height.

- The fluctuations of $\Delta(t_n)$ around $(1 - m)n$ are of order $n^{2^\theta}$.
- For $i \geq 0$, $\mathbb{P}[|u_*(t_n)| = i] \xrightarrow{n \to \infty} (1 - m)m^i$. This is not an immediate consequence of the local convergence!

Recall the local convergence of $t_n$ to

\[ \mathbb{P}(S = i) = (1 - m)m^i \]
Let $u_\star(t_n)$ be the vertex of maximal degree, $\Delta(t_n)$ its degree and $|u_\star(t_n)|$ its height.

- The fluctuations of $\Delta(t_n)$ around $(1 - m)n$ are of order $n^{2^\theta}$.
- For $i \geq 0$, $\mathbb{P}[|u_\star(t_n)| = i] \xrightarrow{n \to \infty} (1 - m)m^i$. This is not an immediate consequence of the local convergence!
- For every sequence $(\lambda_n)_{n \geq 1}$ such that $\lambda_n \to +\infty$:
  $$\mathbb{P} \left[ \left| \mathcal{H}(t_n) - \frac{\ln(n)}{\ln(1/m)} \right| \leq \lambda_n \right] \xrightarrow{n \to \infty} 1.$$
IV. Extensions
Conjecture

We have:

$$\mathbb{E}[\mathcal{H}(t_n)] \sim \frac{\ln(n)}{\ln(1/m)}.$$
Conjecture

We have:

\[ \mathbb{E} \left[ \mathcal{H}(t_n) \right] \xrightarrow{n \to \infty} \frac{\ln(n)}{\ln(1/m)}. \]

Question

What happens when \( \mu \) is any non-generic probability distribution?
Conjecture
We have:
\[ E[\mathcal{H}(t_n)] \sim \frac{\ln(n)}{\ln(1/m)} \quad \text{as} \quad n \to \infty. \]

Question
What happens when \( \mu \) is any non-generic probability distribution?
Theorem (K. 12')

Let \( (r_n)_{n \geq 1} \) be a sequence of positive real numbers.

(i) If \( r_n / \ln(n) \to \infty \), then \( (C_{\text{2nt}}(t) / r_n, 0 \leq t \leq 1) \) converges to the function equal to 0 on \([0, 1]\) as \( n \to \infty \).

(ii) Otherwise, the sequence \( (C_{\text{2nt}}(t) / r_n, 0 \leq t \leq 1) \) is not tight in the space \( C([0, 1], \mathbb{R}) \).
Contour function of $t_n$

**Theorem (K. 12’)**

*Let $(r_n)_{n \geq 1}$ be a sequence of positive real numbers.*

![Graph showing the contour function of $t_n$]
Contour function of $t_n$

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Let $(r_n)_{n \geq 1}$ be a sequence of positive real numbers.

(i) If $r_n / \ln(n) \to \infty$, then $(C_{2nt}(t_n)/r_n, 0 \leq t \leq 1)$ converges to the function equal to 0 on $[0, 1]$ as $n \to \infty$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{contour_function}
\caption{Contour function for $t_n$}
\end{figure}
**Contour function of $t_n$**

**Theorem (K. 12’)**

*Let $(r_n)_{n \geq 1}$ be a sequence of positive real numbers.*

(i) *If $r_n/\ln(n) \to \infty$, then $(C_{2n}(t_n)/r_n, 0 \leq t \leq 1)$ converges to the function equal to 0 on $[0, 1]$ as $n \to \infty$.*

(ii) *Otherwise, the sequence $(C_{2n}(t_n)/r_n, 0 \leq t \leq 1)$ is not tight in the space $C([0, 1], \mathbb{R})$.***