Random planar maps & growth-fragmentations

Igor Kortchemski
(joint work with J. Bertoin and N. Curien)

CNRS & École polytechnique
Motivation

What does a “typical” random surface look like?
Idea: construct a (two-dimensional) random surface as a limit of random discrete surfaces.
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Consider $n$ triangles, and glue them uniformly at random in such a way to get a surface homeomorphic to a sphere.
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Consider \( n \) triangles, and glue them uniformly at random in such a way to get a surface homeomorphic to a sphere.

Figure: A large random triangulation (simulation by Nicolas Curien)
**The Brownian map**

**Problem** (*Schramm* at ICM ’06): Let $T_n$ be a random uniform triangulation of the sphere with $n$ triangles.
The Brownian map

Problem (Schramm at ICM '06): Let \( T_n \) be a random uniform triangulation of the sphere with \( n \) triangles. View \( T_n \) as a compact metric space, by equipping its vertices with the graph distance.
The Brownian map

Problem (Schramm at ICM ’06): Let $T_n$ be a random uniform triangulation of the sphere with $n$ triangles. View $T_n$ as a compact metric space, by equipping its vertices with the graph distance. Show that $n^{-1/4} \cdot T_n$ converges towards a random compact metric space (the Brownian map).
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Solved by *Le Gall* in 2011.
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Since, many different models of discrete surfaces have been shown to converge to the Brownian map (Miermont, Beltran & Le Gall, Addario-Berry & Albenque, Bettinelli & Jacob & Miermont, Abraham), using various techniques (in particular bijective codings by labelled trees).
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(see Le Gall’s proceeding at ICM ’14 for more information and references)
Other motivations:

- links with two dimensional Liouville Quantum Gravity (David, Duplantier, Garban, Kupianen, Maillard, Miller, Rhodes, Sheffield, Vargas, Zeitouni) c.f. the talks of Jason Miller, Scott Sheffield and Vincent Vargas.

- study of random planar maps decorated with statistical physics models (Angel, Berestycki, Borot, Bouttier, Guitter, Chen, Curien, Gwynne, K., Laslier, Mao, Ray, Sheffield, Sun, Wilson), c.f. the talk by Gourab Ray.
Outline

I. **Boltzmann triangulations with a boundary**

II. **Peeling explorations**

III. **Cycles & growth-fragmentations**
I. Boltzmann triangulations with a boundary

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Triangulations
**Definitions**

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A **map** is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations). A map is a **triangulation** when all the faces are triangles. A map is **rooted** when an oriented edge is distinguished.

![Two identical triangulations](image)

**Figure:** Two identical triangulations.
Definitions

A map is a finite connected graph properly embedded in the sphere (up to orientation preserving continuous deformations). A map is a triangulation when all the faces are triangles. A map is rooted when an oriented edge is distinguished.

Figure: Two identical rooted triangulations.
Triangulations with a boundary
Definitions

A **triangulation with a boundary** is a map where all faces are triangles, except possibly the one to the right of the root edge, called the **external face**.
Another example of a triangulation with a boundary
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A **triangulation of the p-gon** is a triangulation with a simple boundary of length $p$. 
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A triangulation with a boundary is a map where all faces are triangles, except possibly the one to the right of the root edge, called the external face.

A triangulation of the $p$-gon is a triangulation with a simple boundary of length $p$.

A triangulation of the $p$-gon chosen at random proportionally to

$$(12\sqrt{3})^{-\#(\text{internal vertices})}$$

is called a (critical) Boltzmann triangulation of the $p$-gon.
Cycles at heights
Let $T^{(p)}$ be a random Boltzmann triangulation of the $p$-gon
**The goal**

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The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the $p$-gon, $B_r(T^{(p)})$ its ball of radius $r$, and

$$L^{(p)}(r) := \left( L_1^{(p)}(r), L_2^{(p)}(r), \ldots \right).$$

be the lengths (or perimeters) of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.
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I. Boltzmann triangulations with a boundary

II. Peeling explorations

III. Cycles & growth-fragmentations
Geometry of random maps

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– **peeling process**, which is an algorithmic procedure that explores a map step-by-step in a Markovian way (Watabiki ’95, Angel ’03).
**Branching peeling explorations**

Intuitively speaking, the **branching peeling process** of a triangulation $t$ is a way to iteratively explore $t$ starting from its boundary and by discovering at each step a new triangle by *peeling an edge* determined by a peeling algorithm $\mathcal{A}$. 
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![Diagram of branching peeling process](image-url)
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The goal

Let $T^{(p)}$ be a random Boltzmann triangulation of the $p$-gon, $B_r(T^{(p)})$ its ball of radius $r$, and

$$L^{(p)}(r) := \left( L_1^{(p)}(r), L_2^{(p)}(r), \ldots \right).$$

be the lengths (or perimeters) of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

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∧→ **Goal**: obtain a functional invariance principle for $(L^{(p)}(r); r \geq 0)$. 

\[ \quad \]
Following the locally largest cycle

\[ \forall r \in \mathbb{N}, \quad \widetilde{L}^{(4)}(0) = 4 \]
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\[ \sim \quad \text{Idea: follow the locally largest cycle at each peeling step and consider its length } \sim L^{(p)}(r) \text{ after } r \text{ peeling steps.} \]

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\tilde{L}^{(4)}(0) = 4, \quad \tilde{L}^{(4)}(1) = 5
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Recall that $\tilde{L}(p)(r)$ the length of the locally largest cycle after $r$ peeling steps.
Scaling limit for the locally largest cycle

Recall that $\tilde{L}(p)(r)$ the length of the locally largest cycle after $r$ peeling steps.

Key fact: $(\tilde{L}(p)(r); r \geq 0)$ is a Markov chain on the nonnegative integers, started at $p$, absorbed at 0 and with explicit transitions.
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Recall that \( \tilde{L}^{(p)}(r) \) the length of the locally largest cycle after \( r \) peeling steps.

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If \( L^{(p)}(r) \) the length of the locally largest cycle at height \( r \), with the help of Bertoin & K. '14 and Curien & Le Gall '14, we get that:

\[ \left( \frac{1}{p} L^{(p)}(\lceil \sqrt{p} \cdot t \rceil); t \geq 0 \right) \xrightarrow{p \to \infty} \left( X \left( \frac{3}{2\sqrt{\pi}} \cdot t \right); t \geq 0 \right), \]

where \( X \) is a càdlàg self-similar process with \( X(0) = 1 \) and absorbed at 0.
The self-similar process $X$
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Let $\xi$ be the spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1 - x)) (x(1 - x))^{-5/2} \, dx,$$

so that $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for every $t \geq 0$ and $q \geq 0$. 
The self-similar process $X$

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Finally, set

$$X(t) = \exp(\xi(\tau(t))) , \quad t \geq 0$$
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Then set

$$\tau(t) = \inf \left\{ u \geq 0; \int_{0}^{u} e^{\xi(s)/2} ds > t \right\}, \quad t \geq 0$$

with the convention that $\inf \emptyset = \infty$, i.e. $\tau(t) = \infty$ whenever $t \geq \int_{0}^{\infty} e^{\xi(s)/2} \, ds$.

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(with the convention $\exp(\xi(\infty)) = 0$), which is a self-similar Markov process (Lamperti transformation).
Defining the growth-fragmentation

We use $X$ to define a self-similar growth-fragmentation process with binary dislocations.
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- Start at time $0$ from a single cell with size $1$, and suppose that its size evolves according to $X$. We interpret each (negative) jump of $X$ as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time $t$ into a mother cell and a daughter.

By Bertoin '15, for every $t \geq 0$, the family of the sizes of cells which are present in the system at time $t$ is cube-summable, and can therefore be ranked in non-increasing order. This yields a random variable with values in $\ell^3$ which we denote by $X(t) = (X_1(t), X_2(t), \ldots)$. 

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$\Rightarrow$ After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size $y$. 

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After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size $y$ and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process $X$ (starting of course from $y$)
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After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size $y$ and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process $X$ (starting of course from $y$), and is independent of the processes of all the other daughter particles.
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And so on for the granddaughters, then great-granddaughters, ...
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Defining the growth-fragmentation

We use $X$ to define a self-similar growth-fragmentation process with binary dislocations. We view $X(t)$ as the size of a typical particle or cell at age $t$, and:

- Start at time 0 from a single cell with size 1, and suppose that its size evolves according to $X$. We interpret each (negative) jump of $X$ as a division event for the cell, in the sense that whenever $\Delta X(t) = X(t) - X(t-) = -y < 0$, the cell divides at time $t$ into a mother cell and a daughter.

After the splitting event, the mother cell has size $X(t)$ and the daughter cell has size $y$ and the evolution of the daughter cell is then governed by the law of the same self-similar Markov process $X$ (starting of course from $y$), and is independent of the processes of all the other daughter particles.

And so on for the granddaughters, then great-granddaughters, ...

By Bertoin ’15, for every $t \geq 0$, the family of the sizes of cells which are present in the system at time $t$ is cube-summable, and can therefore be ranked in non-increasing order. This yields a random variable with values in $\ell_3^{-}\uparrow$ which we denote by $X(t) = (X_1(t), X_2(t), \ldots)$.
**Description of the growth-fragmentation**

We can think of $X$ as a self-similar compensated fragmentation, in the sense that it describes the evolution of particles that grow and divide independently one of the other as time passes:
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$\checkmark$ The dislocations occurring in $X$ are binary, i.e. they correspond to replacing some mass $m$ in the system by two smaller masses $m_1$ and $m_2$ with $m_1 + m_2 = m$. 
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$\downarrow \Rightarrow$ We have $\int (1-x)^2 \nu(dx) < \infty$, but $\int (1-x) \nu(dx) = \infty$ which underlines the necessity of compensating the dislocations.
Cycles and growth-fragmentations

Recall that $L^{(p)}(r) = \left( L_1^{(p)}(r), L_2^{(p)}(r), \ldots \right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.
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**Theorem** (Bertoin, Curien, K. ’15).

We have

$$\left( \frac{1}{p} \cdot L^{(p)}(r \sqrt{p}); r \geq 0 \right) \xrightarrow{(d)} \left( X \left( \frac{3}{2 \sqrt{\pi}} \cdot r \right); r \geq 0 \right),$$

where $X = (X(t); t \geq 0)$ is a self-similar growth-fragmentation process with index $-1/2$ associated with $\xi$. The convergence holds in distribution in the space of càdlàg process taking values in $\ell_{\downarrow}$ equipped with the Skorokhod topology.
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Recall that

$$\Psi(q) = -\frac{8}{3} q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left( x(1-x) \right)^{-5/2} dx.$$
Cycles and growth-fragmentations

Figure: An artistic representation of the cycle lengths of a Boltzmann triangulation with a large boundary obtained by slicing it at all heights: horizontal line segments correspond to the lengths of the cycles of the ball of radius $r$ of the triangulation as $r$ increases. Here the longest cycles are the darkest ones.