**IMO Geomety Problems**

- (IMO 1999/1) Determine all finite sets $S$ of at least three points in the plane which satisfy the following condition:

  for any two distinct points $A$ and $B$ in $S$, the perpendicular bisector of the line segment $AB$ is an axis of symmetry for $S$.

- (IMO 1999/5) Two circles $G_1$ and $G_2$ are contained inside the circle $G$, and are tangent to $G$ at the distinct points $M$ and $N$, respectively. $G_1$ passes through the center of $G_2$. The line passing through the two points of intersection of $G_1$ and $G_2$ meets $G$ at $A$ and $B$. The lines $MA$ and $MB$ meet $G_1$ at $C$ and $D$, respectively. Prove that $CD$ is tangent to $G_2$.

- (IMO 1998/1) In the convex quadrilateral $ABCD$, the diagonals $AC$ and $BD$ are perpendicular and the opposite sides $AB$ and $DC$ are not parallel. Suppose that the point $P$, where the perpendicular bisectors of $AB$ and $DC$ meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles $ABP$ and $CDP$ have equal areas.

- (IMO 1998/5) Let $I$ be the incenter of triangle $ABC$. Let the incircle of $ABC$ touch the sides $BC$, $CA$, and $AB$ at $K$, $L$, and $M$, respectively. The line through $B$ parallel to $MK$ meets the lines $LM$ and $LK$ at $R$ and $S$, respectively. Prove that angle $RIS$ is acute.

- (IMO 1997/2) The angle at $A$ is the smallest angle of triangle $ABC$. The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$. The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$. Show that

  $$AU = TB + TC.$$

- (IMO 1996/2) Let $P$ be a point inside triangle $ABC$ such that

  $$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

  Let $D, E$ be the incenters of triangles $APB, APC$, respectively. Show that $AP, BD, CE$ meet at a point.

- (IMO 1996/5) Let $ABCDEF$ be a convex hexagon such that $AB$ is parallel to $DE$, $BC$ is parallel to $EF$, and $CD$ is parallel to $FA$. Let $R_A, R_C, R_E$ denote the circumradii of triangles $FAB, BCD, DEF$, respectively, and let $P$ denote the perimeter of the hexagon. Prove that

  $$R_A + R_C + R_E \geq \frac{P}{2}.$$

- (IMO 1995/1) Let $A, B, C, D$ be four distinct points on a line, in that order. The circles with diameters $AC$ and $BD$ intersect at $X$ and $Y$. The line $XY$ meets $BC$ at $Z$. Let $P$ be a point on the line $XY$ other than $Z$. The line $CP$ intersects the circle with diameter $AC$ at $C$ and $M$, and the line $BP$ intersects the circle with diameter $BD$ at $B$ and $N$. Prove that the lines $AM, DN, XY$ are concurrent.

- (IMO 1995/5) Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose $G$ and $H$ are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

- (IMO 1994/2) $ABC$ is an isosceles triangle with $AB = AC$. Suppose that

  $M$ is the midpoint of $BC$ and $O$ is the point on the line $AM$ such that $OB$ is perpendicular to $AB$;

  $Q$ is an arbitrary point on the segment $BC$ different from $B$ and $C$;

  $E$ lies on the line $AB$ and $F$ lies on the line $AC$ such that $E$, $Q$, $F$ are distinct and collinear.

- (IMO 1993/2) Let $D$ be a point inside acute triangle $ABC$ such that $\angle ADB = \angle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

  (a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.

  (b) Prove that the tangents at $C$ to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.

- (IMO 1993/4) For three points $P, Q, R$ in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.) Prove that for points $A, B, C, X$ in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

- (IMO 1992/4) In the plane let $C$ be a circle, $L$ a line tangent to the circle $C$, and $M$ a point on $L$. Find the locus of all points $P$ with the following property: there exists two points $Q, R$ on $L$ such that $M$ is the midpoint of $QR$ and $C$ is the inscribed circle of triangle $PQR$.

- (IMO 1991/1) Given a triangle $ABC$, let $I$ be the center of its inscribed circle. The internal bisectors of the angles $A, B, C$ meet the opposite sides in $A', B', C'$ respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

- (IMO 1991/5) Let $ABC$ be a triangle and $P$ an interior point of $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

- (IMO 1990/1) Chords $AB$ and $CD$ of a circle intersect at a point $E$ inside the circle. Let $M$ be an interior point of the segment $EB$. The tangent line at $E$ to the circle through $D$, $E$, and $M$ intersects the lines $BC$ and $AC$ at $F$ and $G$, respectively. If
$$\frac{AM}{AB} = t,$$
find
$$\frac{EG}{EF}$$
in terms of $t$.

- (IMO 1989/2) In an acute-angled triangle $ABC$ the internal bisector of angle $A$ meets the circumcircle of the triangle again at $A_1$. Points $B_1$ and $C_1$ are defined similarly. Let $A_0$ be the point of intersection of the line $AA_1$ with the external bisectors of angles $B$ and $C$. Points $B_0$ and $C_0$ are defined similarly. Prove that:

  (i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.

  (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle $ABC$.

- (IMO 1989/4) Let $ABCD$ be a convex quadrilateral such that the sides $AB$, $AD$, $BC$ satisfy $AB = AD+BC$. There exists a point $P$ inside the quadrilateral at a distance $h$ from the line $CD$ such that $AP = h + AD$ and $BP = h + BC$. Show that
$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

- (IMO 1988/1) Consider two coplanar circles of radii $R$ and $r$ $(R > r)$ with the same center. Let $P$ be a fixed point on the smaller circle and $B$ a variable point on the larger circle. The line $BP$ meets the larger circle again at $C$. The perpendicular $l$ to $BP$ at $P$ meets the smaller circle again at $A$. (If $l$ is tangent to the circle at $P$ then $A = P$.)

  (i) Find the set of values of $BC^2 + CA^2 + AB^2$.

  (ii) Find the locus of the midpoint of $BC$.

- (IMO 1988/5) $ABC$ is a triangle right-angled at $A$, and $D$ is the foot of the altitude from $A$. The straight line joining the incenters of the triangles $ABD$, $ACD$ intersects the sides $AB$, $AC$ at the points $K$, $L$ respectively. $S$ and $T$ denote the areas of the triangles $ABC$ and $AKL$ respectively. Show that $S \geq 2T$.

- (IMO 1987/2) In an acute-angled triangle $ABC$ the interior bisector of the angle $A$ intersects $BC$ at $L$ and intersects the circumcircle of $ABC$ again at $N$. From point $L$ perpendiculars are drawn to $AB$ and $AC$, the feet of these perpendiculars being $K$ and $M$ respectively. Prove that the quadrilateral $AKNM$ and the triangle $ABC$ have equal areas.

- (IMO 1986/2) A triangle $A_1A_2A_3$ and a point $P_0$ are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a set of points $P_1$, $P_2$, $P_3$, $\ldots$, such that $P_{k+1}$ is the image of $P_k$ under a rotation with center $A_{k+1}$ through angle 120° clockwise (for $k = 0, 1, 2, \ldots$). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

- (IMO 1986/4) Let $A$, $B$ be adjacent vertices of a regular $n$-gon ($n \geq 5$) in the plane having center at $O$. A triangle $XYZ$, which is congruent to and initially conincides with $OAB$, moves in the plane in such a way that $Y$ and $Z$ each trace out the whole boundary of the polygon, $X$ remaining inside the polygon. Find the locus of $X$.

- (IMO 1985/5) A circle with center $O$ passes through the vertices $A$ and $C$ of triangle $ABC$ and intersects the segments $AB$ and $BC$ again at distinct points $K$ and $N$, respectively. The circumscribed circles of the triangles $ABC$ and $KBN$ intersect at exactly two distinct points $B$ and $M$. Prove that angle $OMB$ is a right angle.

- (IMO 1984/4) Let $ABCD$ be a convex quadrilateral such that the line $CD$ is a tangent to the circle on $AB$ as diameter. Prove that the line $AB$ is a tangent to the circle on $CD$ as diameter if and only if the lines $BC$ and $AD$ are parallel.

- (IMO 1983/2) Let $A$ be one of the two distinct points of intersection of two unequal coplanar circles $C_1$ and $C_2$ with centers $O_1$ and $O_2$, respectively. One of the common tangents to the circles touches $C_1$ at $P_1$ and $C_2$ at $P_2$, while the other touches $C_1$ at $Q_1$ and $C_2$ at $Q_2$. Let $M_1$ be the midpoint of $P_1Q_1$,and $M_2$ be the midpoint of $P_2Q_2$. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

- (IMO 1982/2) A non-isosceles triangle $A_1A_2A_3$ is given with sides $a_1, a_2, a_3$ ($a_i$ is the side opposite $A_i$). For all $i = 1, 2, 3, M_i$ is the midpoint of side $a_i$, and $T_i$. is the point where the incircle touches side $a_i$. Denote by $S_i$ the reflection of $T_i$ in the interior bisector of angle $A_i$. Prove that the lines $M_1S_1, M_2S_2$, and $M_3S_3$ are concurrent.

- (IMO 1981/1) $P$ is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from $P$ to the lines $BC, CA, AB$ respectively. Find all $P$ for which
$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$
is least.

- (IMO 1981/5) Three congruent circles have a common point $O$ and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point $O$ are collinear.

- (IMO 1979/3) Two circles in a plane intersect. Let $A$ be one of the points of intersection. Starting simultaneously from $A$ two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point $P$ in the plane such that, at any time, the distances from $P$ to the moving points are equal.

- (IMO 1979/4) Given a plane $\pi$, a point $P$ in this plane and a point $Q$ not in $\pi$, find all points $R$ in $\pi$ such that the ratio $(QP + PR)/QR$ is a maximum.

- (IMO 1978/2) $P$ is a given point inside a given sphere. Three mutually perpendicular rays from $P$ intersect the sphere at points $U, V$, and $W; Q$ denotes the vertex diagonally opposite to $P$ in the parallelepiped determined by $PU, PV$, and $PW$. Find the locus of $Q$ for all such triads of rays from $P$

- (IMO 1978/4) In triangle $ABC, AB = AC$. A circle is tangent internally to the circumcircle of triangle $ABC$ and also to sides $AB, AC$ at $P, Q$, respectively. Prove that the midpoint of segment $PQ$ is the center of the incircle of triangle $ABC$.

- (IMO 1977/1) Equilateral triangles $ABK, BCL, CDM, DAN$ are constructed inside the square $ABCD$. Prove that the midpoints of the four segments $KL, LM, MN, NK$ and the midpoints of the eight segments $AKBK, BL, CL, CM, DM, DN, AN$ are the twelve vertices of a regular dodecagon.

- (IMO 1976/1) In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

- (IMO 1975/3) On the sides of an arbitrary triangle $ABC$, triangles $ABR, BCP, CAQ$ are constructed externally with $\angle CBP = \angle CAQ = 45°, \angle BCP = \angle ACQ = 30°, \angle ABR = \angle BAR = 15°$. Prove that $\angle QRP = 90°$ and $QR = RP$.

- (IMO 1974/2) In the triangle $ABC$, prove that there is a point $D$ on side $AB$ such that $CD$ is the geometric mean of $AD$ and $DB$ if and only if

$$\sin A \sin B \leq \sin^2 \frac{C}{2}.$$

- (IMO 1972/2) Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into $n$ quadrilaterals each of which is inscribable in a circle.

- (IMO 1971/4) All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: $X$ is a point on edge $AB$ distinct from $A$ and $B$; similarly, $Y, Z, T$ are interior points of edges $BCCD, DA$, respectively. Prove:

  (a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.

  (b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2AC \sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

- (IMO 1970/1) Let $M$ be a point on the side $AB$ of $\Delta ABC$. Let $r_1, r_2$ and $r$ be the radii of the inscribed circles of triangles $AMC, BMC$ and $ABC$. Let $q_1, q_2$ and $q$ be the radii of the escribed circles of the same triangles that lie in the angle $ACB$. Prove that
$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

- (IMO 1970/5) In the tetrahedron $ABCD$, angle $BDC$ is a right angle. Suppose that the foot $H$ of the perpendicular from $D$ to the plane $ABC$ is the intersection of the altitudes of $\Delta ABC$. Prove that
$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$
For what tetrahedra does equality hold?

- (IMO 1969/4) A semicircular arc $\gamma$ is drawn on $AB$ as diameter. $C$ is a point on $\gamma$ other than $A$ and $B$, and $D$ is the foot of the perpendicular from $C$ to $AB$. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line $AB$. Of these, $\gamma_1$ is inscribed in $\Delta ABC$, while $\gamma_2$ and $\gamma_3$ are both tangent to $CD$ and to $\gamma$, one on each side of $CD$. Prove that $\gamma_1, \gamma_2$ and $\gamma_3$ have a second tangent in common.

- (IMO 1968/4) Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

- (IMO 1967/1) Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\angle BAD = \alpha$. If $\Delta ABD$ is acute, prove that the four circles of radius 1 with centers $A, B, C, D$ cover the parallelogram if and only if
$$a \leq \cos\alpha + \sqrt{3}\sin\alpha.$$

- (IMO 1967/4) Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles $ABC$ that are similar to $\Delta A_1B_1C_1$ (so that vertices $A_1, B_1, C_1$ correspond to vertices $A, B, C$, respectively) and circumscribed about triangle $A_0B_0C_0$ (where $A_0$ lies on $BC$, $B_0$ on $CA$, and $AC_0$ on $AB$). Of all such possible triangles, determine the one with maximum area, and construct it.

- (IMO 1966/3) The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

- (IMO 1965/3) Given the tetrahedron $ABCD$ whose edges $AB$ and $CD$ have lengths $a$ and $b$ respectively. The distance between the skew lines $AB$ and $CD$ is $d$, and the angle between them is $\omega$. Tetrahedron $ABCD$ is divided into two solids by plane $\varepsilon$, parallel to lines $AB$ and $CD$. The ratio of the distances of $\varepsilon$ from $AB$ and $CD$ is equal to $k$. Compute the ratio of the volumes of the two solids obtained.

- (IMO 1965/5) Consider $\Delta OAB$ with acute angle $AOB$. Through a point $M \neq O$ perpendiculars are drawn to $OA$ and $OB$, the feet of which are $P$ and $Q$

respectively. The point of intersection of the altitudes of $\Delta OPQ$ is $H$. What is the locus of $H$ if $M$ is permitted to range over (a) the side $AB$, (b) the interior of $\Delta OAB$?

- (IMO 1964/3) A circle is inscribed in triangle $ABC$ with sides $a, b, c$. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from $\Delta ABC$. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of $a, b, c$).

- (IMO 1964/6) In tetrahedron $ABCD$, vertex $D$ is connected with $D_0$ the centroid of $\Delta ABC$. Lines parallel to $DD_0$ are drawn through $A, B$ and $C$. These lines intersect the planes $BCD, CAD$ and $ABD$ in points $A_1, B_1$ and $C_1$, respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point $D_0$ is selected anywhere within $\Delta ABC$?

- (IMO 1963/2) Point $A$ and segment $BC$ are given. Determine the locus of points in space which are vertices of right angles with one side passing through $A$, and the other side intersecting the segment $BC$.

- (IMO 1962/5) On the circle $K$ there are given three distinct points $A, B, C$. Construct (using only straightedge and compasses) a fourth point $D$ on $K$ such that a circle can be inscribed in the quadrilateral thus obtained.

- (IMO 1961/4) Consider triangle $P_1P_2P_3$ and a point $P$ within the triangle. Lines $P_1P, P_2P, P_3P$ intersect the opposite sides in points $Q_1, Q_2, Q_3$ respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is $\leq 2$ and at least one is $\geq 2$.

- (IMO 1961/5) Construct triangle $ABC$ if $AC = b, AB = c$ and $\angle AMB = \omega$, where $M$ is the midpoint of segment $BC$ and $\omega < 90°$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

- (IMO 1960/3) In a given right triangle $ABC$, the hypotenuse $BC$, of length $a$, is divided into $n$ equal parts ($n$ an odd integer). Let $\alpha$ be the acute angle subtending, from $A$, that segment which contains the midpoint of the hypotenuse. Let $h$ be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan \alpha = \frac{4nh}{(n^2 - 1)a}.$$

- (IMO 1960/5) Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).

(a) Find the locus of the midpoints of segments $XY$, where $X$ is any point of $AC$ and $Y$ is any point of $B'D'$.

(b) Find the locus of points $Z$ which lie on the segments $XY$ of part (a) with $ZY = 2XZ$.

- (IMO 1959/5) An arbitrary point $M$ is selected in the interior of the segment $AB$. The squares $AMCD$ and $MBEF$ are constructed on the same side of $AB$, with the segments $AM$ and $MB$ as their respective bases. The circles circumscribed about these squares, with centers $P$ and $Q$, intersect at $M$ and also at another point $N$. Let $N'$ denote the point of intersection of the straight lines $AF$ and $BC$.

  (a) Prove that the points $N$ and $N'$ coincide.

  (b) Prove that the straight lines $MN$ pass through a fixed point $S$ independent of the choice of $M$.

  (c) Find the locus of the midpoints of the segments $PQ$ as $M$ varies between $A$ and $B$.

- (IMO 1959/6) Two planes, $P$ and $Q$, intersect along the line $p$. The point $A$ is given in the plane $P$, and the point $C$ in the plane $Q$; neither of these points lies on the straight line $p$. Construct an isosceles trapezoid $ABCD$ (with $AB$ parallel to $CD$) in which a circle can be inscribed, and with vertices $B$ and $D$ lying in the planes $P$ and $Q$ respectively.